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THE GEOMETRICIAN:

CONTAINING

ESSAYS

ON

PLANE GEOMETRY,

AND

TRIGONOMETRY:

With their Application to the SOLUTIONS of a
VARIETY of PROBLEMS, which are of great Uſe
in *meaſuring Heights* and *Diſtances* of Places, ſur-
veying of *Counties*, *Sea-Coaſts*, *&c.*

BY BENJAMIN DONN,

Teacher of the *Mathematics*, and Lecturer in *experimental Philoſophy*,

Author of the Eſſays on *Arithmetic*, *Book-keeping*, the *Britiſh Mariner's Aſſiſtant*, *&c.*

The SECOND EDITION.

Quicquid in aſtronomicis, geographicis, vel nauticis, efficiendum, geometriæ et trigonometriæ attribuendum eſt.

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TO THE
TUTORS IN THE UNIVERSITIES,
MASTERS OF ACADEMIES,
AND OTHER
TEACHERS OF GEOMETRY,
IN GREAT-BRITAIN.

GENTLEMEN,

IT is probable that many of you, as well as myſelf, have found it frequently difficult to initiate young Gentlemen into a neceſſary Acquaintance with Geometry by the Elements of *Euclid*, and therefore have wiſhed for a more eaſy Introduction to that valuable Science, which, at the ſame time as it facilitated the Attainment of Geometry, might not depart (as many do) too much from the geometrical Spirit of the Ancients, which is ſo neceſſary for acquiring a Habit of reaſoning with Propriety and Judgement.

Such a Treatiſe, with its Application to Trigonometry, I have now attempted, and beg Leave to offer to the Public, under your Protection; and, if it ſhould be ſo far approved of, by you, as to be put into the Hands of your Pupils, I flatter myſelf you will find the Science will be acquired by them in much leſs Time than is uſual, and with greater Eaſe to yourſelves. If this meets with your Encouragement, it will induce me to purſue my original Deſign, of preſenting the Public with other Eſſays, for rendering the ſeveral Branches of mathematical Literature, both ancient and modern, more pleaſant in the Study, and more eaſily to be attained, than by any other Courſe, hitherto publiſhed, in our Language.

I am, Gentlemen,

Your humble Servant,

B. DONN.

Kingſton, near Taunton,
July 5, 1775.

THE PREFACE.

IN the general Preface to the *Eſſays on Arithmetic*, publiſhed in the Year 1758, the Public were acquainted with our Deſign of preſenting them with a new Courſe of mathematical Learning, a Thing generally allowed to be much wanted: * For the great and numerous Improvements, which have been made ſince any Courſe has been publiſhed in the *Engliſh* Language, make a new Courſe abſolutely neceſſary. For, as thoſe Sciences, with their Improvements, are diſperſed in a Multitude of Authors, the young Student knows not how to proceed; and, even to a Maſter, it is no eaſy Taſk to direct: Whereas, if theſe Sciences are brought into a regular Courſe, as they depend on each other, the Student will learn with more Eaſe, Pleaſure, and Diſpatch.

---

* "Mathematical Learning, during the laſt and preſent Centuries, has made a moſt ſurprizing Progreſs; and Truth, aſſiſted by the uncontroverted Principles of this Science, has baniſhed hypothetical Chicanery from the Regions of Philoſophy. It is, therefore, no Wonder that a great Variety of Authors, deſirous of extending ſo valuable a Branch of Science, ſhould have written on every Part of mathematical Learning. But ſtill a Courſe of Mathematics and Natural-Philoſophy, tracing the Science from its firſt Principles, and exhibiting the Demonſtrations on which each Rule or Problem is founded, is ſtill wanting; there being none, in our own Language, that can, with any Show of Juſtice, be called a Courſe of Mathematics and Natural-Philoſophy, according to the modern Improvements, and properly adapted to Learners. This Defect Mr. *Donn* has undertaken to ſupply." MONTHLY REVIEW for JULY, 1758.

As this, if well executed, is evidently a Work of public Utility, it is hoped all mathematical Maſters and Lovers of thoſe Sciences will promote it in ſuch a Degree as its Uſefulneſs deſerves. In Proſecution of which Deſign, this Volume is offered for public Acceptance, containing Eſſays on *Logarithms*, *Plane Geometry* and *Trigonometry*.

It may, perhaps, be objected, by ſome, that, ſince the firſt Edition of this Volume was printed, ſeveral Volumes of a Courſe have been publiſhed, by an ingenious Gentleman, and that therefore a new Courſe of the Mathematics is not now neceſſary. But I beg Leave to obſerve, that the Maſters of Academies, and other Teachers, muſt be ſenſible that *that* Courſe is not drawn up in a Manner proper to introduce Learners to thoſe difficult Sciences, as, in ſeveral of the Volumes, the difficult Parts of the Subjects are treated of before the more eaſy, *&c.* Of this the learned Author was ſo ſenſible, that, on my acquainting him (when I had the Pleaſure of ſeeing him in *London*) that I obtained my firſt Knowledge of Fluxions from his excellent Treatiſe, without the Aſſiſtance of a Maſter, when I was about 18 Years of Age; he anſwered, with a Kind of Surprize, "From *my* Fluxions! My Trea-"tiſe was intended for *Maſters*, not for *Learners*!" I have not made this Remark either out of any high Opinion I entertain of my own Abilities or to depreciate his Works, but only to give his own Idea of his Publications. I eſteem him as an excellent Mathematician, and ſhall therefore only obſerve farther, that, though it would perhaps be thought Preſumption in me to promiſe to treat of the ſame Subjects in a more maſterly Manner, yet, I may venture to ſay, I hope to give them on a much more enlarged Plan, with reſpect to practical Mathematics, and more eaſy to be underſtood by young Mathematicians.

Notwithſtanding it is intended to treat of the ſeveral Sciences in ſucceſſive Order, as they ought to be ſtudied, to form a regular Courſe, for the Sake of ſuch Readers as are deſirous of acquiring a general Knowledge therein; yet, at the ſame Time, on Account of Others, who may only be inclined to ſtudy ſuch Parts as may be more particularly uſeful to them, we ſhall endeavour to make each Subject diſtinct and entire by itſelf, ſo that every Perſon may have it at his own Choice to read what Subjects he pleaſes.

As this Work is written as a Courſe, the Reader of any Eſſay is ſuppoſed to know no more of the Mathematics than is contained in the preceding Eſſays, and therefore muſt not expect any Theorems, *&c.* which cannot be demonſtrated by the foregoing; ſo that if he happens to find any valuable Things omitted, in any Eſſay of ours, which he has ſeen in any other Work, we would not have him think we have forgot to inſert it, but that we refer it to a more proper Place, in a future Eſſay. For Inſtance, in Logarithms, we have omitted the Inveſtigation of Series, becauſe it depends on the higher Parts of the Mathematics, and muſt therefore be omitted until thoſe Parts are explained.

Eſſays on Circulating-Decimals and Book-Keeping were prefixed to the firſt Edition of our *Geometrician*, and publiſhed under the Title of *The Accountant and Geometrician*; but, as they have no Connection with Geometry, it was thought beſt to omit them in this Edition: And, as the Eſſays on Book-Keeping have been enlarged in Number, by the Addition of *The Shopkeeper's, Steward's, and Factor's, Companion*, it was thought beſt to publiſh them in a Volume by themſelves, entitled, *The Accountant*; and, inſtead of them, in this Volume, to give the *Elements* ‡ *of Plane*

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‡ To oblige the Purchaſers of the firſt Edition of our *Geometry*, we have printed a larger Number of the *Trigonometry*, that they may not be put to the Expence of purchaſing the whole Volume a ſecond Time; but it will only be ſold ſeparate to thoſe who have already the firſt Edition.

Trigonometry, with their Application to Altimetry and Longimetry. For a more circumſtantial Account of the Contents of this Volume, we beg Leave to refer the Reader to the Titles and Prefaces to the ſeveral Eſſays.

B. DONN.

◆

Juſt publiſhed,
(From the AUTHOR's MANUSCRIPT,)
A SERMON,
Preached at the Funeral of Mr. ABRAHAM DONN,
By the late pious and ingenious
JAMES HERVEY, M. A.
Rector of *Weſton-Favel*, in *Northamptonſhire*, and Author of the MEDITATIONS and CONTEMPLATIONS.

Printed for B. LAW, in AVE-MARIA-LANE.

AN
ESSAY
ON THE
NATURE, INVESTIGATION, and APPLICATION,
OF
LOGARITHMS.

By BENJAMIN DONN.

Si numeris in ratione geometrica progredientibus ſubſcribantur totidem alii æquidifferentes; dicuntur hi illorum Logarithmi.

Wolfius, Elem. Arith.

PREFACE.

THE Invention of *Logarithms* is univerſally allowed to be one of the moſt uſeful Diſcoveries in the Art of Numbers: For, as Dr. *Keil* juſtly obſerves, "It is by their Means "that Numbers almoſt infinite, and ſuch as "are otherwiſe impracticable, are managed "with eaſe and expedition. By their Aſſiſtance "the Mariner ſteers his Veſſel, the Geometri"cian inveſtigates the Nature of the higher "Curves, the Aſtronomer determines the Pla"ces of the Stars, and the Philoſopher accounts "for other Phenomena of Nature; and laſtly, "the Uſurer computes the Intereſt of his Mo"ney."

The Honour of the Invention of Logarithms is ſufficiently ſecured to Lord *John Neper*, Baron of *Merchiſton* in *Scotland*, by the Teſtimonies of all who have treated on the Subject, as well Foreigners, as Inhabitants of *Britain*.

In 1614, Lord *Neper* publiſhed at *Edinburgh* a Canon of Logarithms, in which 0 is taken for the Logarithm of Radius (or 1000000000,) and the Logarithms of the Sines are affirmative.

This Book being in *Latin*, was tranſlated into *Engliſh* by Mr. *Edward Wright*, and after his Deceaſe publiſhed by his Son *Samuel*, in the Year 1716.

The great Uſefulneſs of Logarithms was immediately perceived by the learned Mathematicians of the Age; and amongſt them Mr.

Briggs (then Profeſſor of Geometry at *Greſham* College, *London,)* was ſo charmed with the noble Invention, that he went to *Scotland* on Purpoſe to conſult with the Inventor, and aſſiſt him in perfecting his Deſign: The Reſult of their Conſultation was, that the Inventor and Mr. *Briggs* agreed to change the Form of the Logarithms into a more compendious one; in which the Logarithm of 1 was to be 0; and 1000000, *&c.* the Logarithm of Radius.

In the Year 1619, the noble Inventor being dead, his Son Lord *Robert* publiſhed the Conſtruction of the *admirable Canon*; in the Preface to which he takes Occaſion to mention the great Friendſhip his Father had for Mr. *Briggs*; on whom now the whole Labour of calculating Tables entirely devolved. This did not diſcourage Mr. *Briggs*, for applying himſelf to the Work with a Degree of Induſtry that ſurprized the whole World, he publiſhed at *London*, in 1624, his *Arithmetica Logarithmica*, agreeable to the new Form, for the firſt 20 Chiliads of Logarithms, (or from 1 to 20000,) and for 11 Chiliads more, *viz.* from 90000 to 101000, calculated to 14 Places of Figures. He alſo gave Directions for ſupplying the intermediate Chiliads.

In the Year 1627, *Adrian Vlac* (or *Flack)* publiſhed a Canon to 10 Places of Figures, in which the intermediate Chiliads were filled up, according to Mr. *Briggs*'s Directions.

In 1633, *Vlac* publiſhed *Brigg*'s Logarithms of Sines and Tangents to every 10 Seconds, and of Numbers from 1 to 20000.

PREFACE.

In the Year 1658, Dr. *John Newton* publiſhed at *London* *Trigonometria Britannica*, in which are Tables of Logarithms of Numbers from 1 to 100000, and alſo the Logarithms of Sines and Tangents to $\frac{1}{100}$ Parts of every Degree.

The Editions of the Tables of Logarithms now moſt eſteemed, are *Sherwin*'s, containing the Logarithms taken from Dr. *John Newton*; and *Gardener*'s, containing the Logarithms of Numbers taken alſo from Dr. *John Newton*, and the Logarithms of Sines and Tangents to every 10 Seconds, taken from *Vlac*'s *Canon Magnus.*

We muſt not however forget Mr. *Dodſon*'s *Antilogarithmic Canon*, publiſhed in the Year 1742; being a Table of Numbers, conſiſting of 11 Places of Figures, correſponding to all Logarithms under 100000; which indeed is a laborious and ingenious Work.

Other Authors who have laboured in compleating Tables, &c. are, in the Year 1619, *Speidell* and *Urſinus*; in 1620, *Gunter*; in 1624, *Wingate*; in 1625, *Urſinus* again, and *Kepler*; in 1626, *Henrion*; in 1627, *Speidell*, and *Kepler* again in his *Rudolphine Tables*; alſo ſometime before this, *Gellibrand*, as in *Vlac*'s *Trigonometria Artificialis*, publiſhed this Year, the explanatory Part is an Abridgement of *Gellibrand*'s Trigonometry. In the Year 1631, *Norwood*; in 1632, *Cavalerius*; in 1633, *Rowe*; in 1634, *Frobenius*; in 1670, *Caramuel:* Since which Time, Dr. *Gregory*, *Mercator*, Sir *Iſaac Newton*, Dr. *Halley*, *William Jones*, Eſq. and a Number of other learned

Men, have publiſhed infinite converging Series, for computing Logarithms more expeditiouſly; ſome of which we may have Occaſion to explain hereafter.

It would be endleſs to give a Liſt of all the Authors who have touched on Logarithms; becauſe, almoſt all who have treated on Trigonometry, on Navigation, &c. have given compendious Tables of Logarithms, &c.

By this compendious ESSAY, it is hoped the Learner may attain as clear Ideas of the Nature and Application of Logarithms, as he could by much larger Treatiſes.

Proper Tables of Logarithms will be given in the Eſſay on *Trigonometry*, or *Navigation*, as they will there be of more immediate Uſe, &c. as the Tables of Sines, Tangents, &c. could not with any Propriety be given in this Place.

Bideford, February 7,
1759.

B. DONN.

A

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AN
ESSAY
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CHAP. I.

Of DEFINITIONS, *and the Nature and Inveſtigation of* LOGARITHMS.

1. LOGARITHMS (λόγος and ἀριθμός) are Numbers ſo contrived and adapted to other Numbers, that by their Addition, or Subtraction, the Product, or Quotient, of the Numbers to which they are adapted, may be found.

This Definition expreſſes the principal Deſign and Uſe of thoſe artificial Numbers, which Mathematicians diſtinguiſh by the Name of *Logarithms.* But here, (as it frequently happens in other Sciences,) a clear, accurate, etymological Senſe, cannot be underſtood, till we have made a particular Enquiry into the Principles of the Science itſelf.

2. To proceed then, let us conſider the Nature of a geometrical Progreſſion, with the Indices ſet over the reſpective Terms.

For Example:

Indices,	0.	1.	2.	3.	4.	5.	6.	*&c.*
Terms,	1.	2.	4.	8.	16.	32.	64.	*&c.*

Here it is evident, that if we add any two Indices together, their Sum will be the Index of that Number, which is equal to the Product of the Numbers whoſe Indices are added together: Thus, *e.g.* The Indices 2 and 3 added together are = 5; the Numbers anſwering to the Indices 2 and 3, are 4 and 8; and 4 × 8 = 32 = the Number anſwering to the Index 5.

It alſo follows from the Nature of ſuch Progreſſions, that if we ſubtract one Index from another, their Difference will be the Index of that Number or Term, which is equal to the Quotient of the two Numbers or Terms, correſponding to the Indices whoſe Differences we found: *e.g.* The Indices 6 — 4 = 2; the Numbers correſponding to theſe Indices are 64 and 16, and 64 ÷ 16 = 4 = the Number correſponding to the Index 2.

3. Hence it follows, that (by Art. 1.) the Indices of a Series of Numbers in geometrical Progreſſion, are Logarithms of the Terms in that Progreſſion.

4. Hence, by the Nature of geometrical Progreſſions, it follows, that, if the Logarithm of any Number be multiplied by the Index of the Power, the Product will be equal to the Logarithm of the Root when involved to the Height denoted by the Index. Thus, for Inſtance: The Index or Logarithm of 4 in the above Series 2, being multiplied by 3 = 6 = the Index or Logarithm of 64; and 64 = 4 cubed = 4 × 4 × 4.

5. From hence alſo it follows, that the Index or Logarithm of any Number being divided by 2, the Quotient will be the Index or Logarithm of the ſquare Root; but if divided by 3, the Index or Logarithm of the cube Root; if divided by 4, the Quotient will be the Index or Logarithm of the 4th Power, *&c.*

Example. The Logarithm of 64 in the above Series is 6, which being divided by 3, the Quotient is 2, the Index or Logarithm of 4, which is the cube Root of 64; for 4 × 4 × 4 = 64.

6. It follows, from the Nature of geometrical Progreſſions, that the Ratio of the firſt, and any other Term, is compoſed of ſo many equal Ratios, as are expreſſed by the Index of that other Term: *e. g.* In the above Series, the Ratio of the firſt and 7th Term is made up of 6 equal Ratios, for the Ratio of the 1ſt to the 2d Term is as 1 to 2, and of the 2d to the 3d as 1 to 2, and of the 3d to the 4th as 1 to 2, of the 4th to the 5th as 1 to 2, of the 5th to the 6th as 1 to 2, and of the 6th to the 7th as 1 to 2; ∴ the Ratio of the 1ſt to the 7th is compoſed of $1 \times 1 \times 1 \times 1 \times 1 \times 1$ to $2 \times 2 \times 2 \times 2 \times 2 \times 2$; *i. e.* as 1 to 64. Hence plainly appears the Propriety of the Term *Logarithms*, it ſignifying, according to its Etymology, a *Number of Ratios.*

7. Logarithms may be of various Kinds; for inſtead of the Series 1, 2, 4, &c. above, we may ſubſtitute any other; thus *e. g.* if we write the geometrical Series 1, 10, 100, &c. inſtead of 1, 2, 4, &c. it will ſtand thus;

Indices or Logarithms	0.	1.	2.	3.	4.	&c.
Terms or natural N^{os}.	1.	10.	100.	1000.	10000,	&c.

This is the Form of *Brigs*'s Logarithms; which are thoſe we ſhall now make it our Buſineſs to diſcourſe on.

The greateſt Difficulty now remaining is to explain the Method made Uſe of by Mr. *Brigs*, for finding the Logarithms of the intermediate Terms, becauſe the natural Numbers, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, &c. are not in a geometrical Progreſſion.

8. Before we proceed any farther, it may be proper to obſerve, that, if we ſuppoſe a geometrical Series beginning from Unity, whoſe common Multiplier is $1 + m$, the Series will be 1, $1 + m$, $\overline{1 + m}|^{2}$, $\overline{1 + m}|^{3}$, $\overline{1 + m}|^{4}$, &c. Now if m be ſup-

poſed indefinitely little, it is evident that ſome of the Terms will approach indefinitely near to the natural Numbers 2, 3, 4, 5, &c, and, conſequently, it is poſſible to find the natural Number anſwering to any Logarithm, very nearly, though not exactly true.

9. The Method of doing which may be thus explained. —— By the Definition of Logarithms, if x and y repreſent any two Numbers, the Logarithm of $\overline{x \times y}$ = the Logarithm of x + the Logarithm of y; and therefore, (by Art. 5.) the Logarithm of $\sqrt{x \times y} = \frac{\text{the Log. } x + \text{Log. } y}{2}$; Hence, the[*] *arithmetical Mean* of any two Logarithms is equal to the Logarithm of the *geometrical Mean* of the two natural Numbers; conſequently, the Logarithm of any intermediate Number may be found, by finding the geometrical Mean between the two given Numbers, (whoſe Logarithms are given,) and the Logarithm anſwering thereto, (*viz.* the arithmetical Mean of the two given Logarithms:) Then, the geometrical Mean between the geometrical Mean juſt found and the neareſt Extreme, and the Logarithm anſwering thereto. After this Manner we proceed, till we find a geometrical Mean ſo very nearly equal to the Number whoſe Logarithm we want, as we ſhall think ſufficiently near the Truth; then the Logarithm of that geometrical Mean is ſufficiently near the Logarithm required.

10. To illuſtrate this, let it be required to find the Logarithm of 9. Here we have the Logarithms of 1 and 10 given.

---

[*] Half the Sum of any two Numbers is called an *arithmetical Mean*; and the ſquare Root of the Product of two Numbers is called the *geometrical Mean*.

(1.) The Log. of 1 = 0
Log. of 10 = 1
Sum = 1
$\frac{1}{2}$
Arithmetical Mean 0.5

And $\sqrt{1 \times 10} = \sqrt{10} = 3.1622777$, ∵ the Log. of 3.1622777 is = 0.5.

(2.) Hence we have the Logarithms of 3.1622777 and 10 given.

Log. of 10 - - = 1.
Log. of 3.1622777 = 0.5
1.5
$\frac{1}{2}$
0.75

Hence the Log. of $\sqrt{10 \times 3.1622777}$, *viz.* of 5.6234132 is 0.75.

(3.) We have now the Logarithms of 10 and 5.6234132 given.

$$\text{Numbers}\left\{\begin{array}{l}10.\ \text{- - - - its Log.} = 1. \\ 5.6234132\ \text{- its Log.} = 0.75\end{array}\right.$$

1.75
$\frac{1}{2}$

$\sqrt{10 \times 5.6234132} = 7.4989421$ its Log. = 0.875

(4.) Here we have the Log. of 10 and 7.4989421 given.

$$\text{Numbers}\left\{\begin{array}{l}10. \\ 7.4989421\end{array}\right\}\text{Their Logarithms}\left\{\begin{array}{l}1. \\ 0.875\end{array}\right.$$

1.875
$\frac{1}{2}$

$\sqrt{10 \times 7.4989421} = 8.6596431$ its Log. = 0.9375

(5.) Hence we have the Logarithms of 10 and 8.6596431 given.

$$\text{Numbers}\left\{\begin{array}{l}10. \\ 8.6596431\end{array}\right\}\text{Their Logarithms}\left\{\begin{array}{l}1. \\ 0.9375\end{array}\right.$$

1.9375
$\frac{1}{2}$

$\sqrt{10 \times 8.6596431} = 9.3057204$ its Log. = 0.96875

(6.) We have now the Log. of 8.6596431 and 9.3057204 given.

$$\left.\begin{matrix}9.3057204\\8.6596431\end{matrix}\right\}\begin{matrix}\text{Their}\\\text{Logarithms}\end{matrix}\left\{\begin{matrix}0.96875\\0.9375\end{matrix}\right.$$

$$\begin{array}{r}\hline 1.90625\\ \frac{1}{2}\ \overline{}\end{array}$$

$$\sqrt{9.3057204 \times 8.6596431} \quad \text{its Log. } 0.953125$$

And proceeding after this Manner, after 25 Extractions the Logarithm of 8.9999998 will be found to be 0.9542425, which we take for the Logarithm of 9, becauſe 8.9999998 differs but $\frac{2}{10000000}$ or $\frac{1}{5000000}$ from 9. —— By the ſame Method the Logarithms of the prime Numbers were found.

11. The Logarithms of the prime Numbers being found, the Logarithms of the Numbers compoſed of them are found by adding the Logarithms of the component Parts; the Reaſon of which is evident from the Definition, in Art. 1. Take an Example, Given the Logarithm of 2 = 0.30103, and the Logarithm of 3 = 0.4771212, to find the Logarithm of 2 × 3 or 6.

$$\begin{array}{rl}\text{The Log. of } 2 = & 0.30103\\ \text{Log. of } 3 = & 0.4771212\\ \hline \text{Log. of } 6 = & 0.7781512\\ \hline\end{array}$$

12. The Logarithm of any Number being given, the Logarithm of any Number equal to the Square, or Cube, &c. of that Number, may be found, by multiplying the Logarithm of the Root by 2, 3, &c. reſpectively, according as the Power, whoſe Logarithm is to be found, is a Square, or Cube, &c. This is manifeſt by Art. 4.

13. Laſtly, the Logarithm of any Power being given, the Logarithm of its Root is found by dividing that Logarithm by the Index of the given Power: By Art. 5.

14. After the Method juſt now deſcribed the Tables of Logarithms were at firſt made. And if we

conſider the great Labour and Patience neceſſary to the Accompliſhment of ſo great a Work, how much are we beholden to Lord *Neper* and Mr. *Brigs*? Certainly their Names will, and ought to be perpetuated from one Age to another, even to the End of the World!

There have lately been invented much eaſier Methods of conſtructing Logarithms than the above; but as they depend on Fluxions, Conic Sections, *&c.* they cannot be underſtood in this Place. And it is ſufficient to have ſhewn the Method by which the Tables were made, as we have now no Occaſion to conſtruct new ones.

15. Having explained the Nature of making a Table of Logarithms of Numbers greater than Unity, the next Thing to be done is to ſhew how the Logarithms of fractional Numbers may be found. In Order to which let it be obſerved, that, as we have hitherto ſuppoſed a geometrical Series to increaſe from an Unit on the right Hand, we may now ſuppoſe it to decreaſe from an Unit towards the left Hand, as here annexed.

Log.	—3.	—2.	—1.	0.	+1.	+2.	+3,	*&c.*
	C.			A.			B.	
Numbers	$\frac{1}{1000}$.	$\frac{1}{100}$.	$\frac{1}{10}$.	1.	10.	100.	1000,	*&c.*

Here the Series from the Point A, or Unity, increaſes towards B, and decreaſes towards C.

It is evident by a bare Inſpection, that, if the Numbers are both Fractions, the Sum of their Logarithms is equal to the Logarithm of their Product. But if one Number is on the Right-hand of the Point A, and the other on the Left of it; that is, if one Number is greater than an Unit, and the other a Fraction, then the Difference of their Logarithms will be equal to the Logarithm of their Product; and the Logarithm of the Product will be on the ſame Side of the Point A, as is the greater of the two Logarithms.

16. In Order to make a proper Diſtinction, we may denote the Logarithm of any Number greater than an Unit by the Sign +, for they may be reckoned Affirmative; and then the fractional Quantities muſt be conſidered as negative.

17. It is alſo evident from Inſpection of the above Series, that if a Number, having an affirmative Logarithm, is divided by another Number whoſe Logarithm is a greater affirmative Number, the Difference of their Logarithms taken negatively will be the Logarithm of their Quotient. Thus, *e. g.* the Logarithm of 100 = 2, and the Logarithm of 1000 = 3; their Difference taken negatively is — 1 = the Logarithm of $\frac{1}{10}$, the Quotient of 100 ÷ 1000. —— This might have been deduced from the Nature of Fractions. For the Numerator of any Fraction may be eſteemed as a Dividend, and the Denominator as a Diviſor: Therefore, by the Nature of Logarithms, (Art. 1.) the Logarithm of the Numerator *minus* the Logarithm of the Denominator is equal to the Logarithm of the Fraction: But ſince the Denominator of a proper Fraction is always greater than the Numerator, its Logarithm will be greater than that of the Numerator, and conſequently it will be impoſſible to take the Logarithm of the Denominator from that of the Numerator; therefore, we are obliged to take the Logarithm of the Numerator from that of the Denominator, and expreſs the Difference negatively, for the Logarithm of the Fraction.

18. The Logarithm of 1, 10, 100, 1000, *&c.* being 0, 1, 2, 3, *&c.* reſpectively, it is manifeſt, that the Logarithms of the intermediate Terms may be conſidered as a mixed Number, made up of an Integer and decimal Fraction; and that the integral Part, called by ſome Writers the *Index*, *(Lat.)* by Others the *Exponent*, (from *expono*, *Lat.*) and by Others the *Characteriſtick*, (from *Character*, *Lat.*) is one leſs than the Number which expreſſes how many Figures the natural Number conſiſts of; *e. g.* if the

natural Number has 4 Places, the Characteriſtick of the Logarithm will be 3; if of 3 Places, 2; if of 2 Places, 1, &c.

19. Diviſion being performed by Subtraction of the Logarithms, and the Logarithm of 10 being 1, it follows, that the Logarithm of $\frac{1}{10}$ of any Number, muſt be one leſs than the Logarithm of that Number. Thus, *e. g.* the Logarithm of 7851 being + 3.894925,

The Log. of	$\frac{7851}{10} = 785.1$	is	$+2.894925$	Affirmative Logarithms.
	$\frac{785.1}{10} = 78.51$		$+1.894925$	
	$\frac{78.51}{10} = 7.851$		$+0.894925$	
	$\frac{7.851}{10} = .7851$		-0.105075	Negative Logarithms.
	$\frac{.7851}{10} = .07851$		-1.105075	
	$\frac{.07851}{10} = .007851$		-2.105075	
	&c.			

The above affirmative Logarithms were found by ſucceſſively ſubtracting the Logarithm of 10, *viz.* + 1 from the preceding Logarithm; then, to find the firſt negative Logarithm, + 1 (the Log. of 10) was to be taken from the Logarithm + 0.894925; but as the Log. to be ſubtracted is greater than that from which it was to be taken, we are obliged to ſubtract the leſſer 0.894925 from the greater 1, and expreſs their Difference negatively, for the required Logarithm, (as directed in Art. 17,) which gives — 0.105075. Then to find the other negative Logarithms, the Logarithm + 1 being to be ſubtracted from a negative Logarithm, we have only to add 1 ſucceſſively, and expreſs the Sum negatively; for by Art. 46, *Eſſay on Arithmetick*, to ſubtract an Affirmative is the ſame as to add a Negative.

20. The Logarithms of Fractions may be expreſſed in ſuch a Manner, that the fractional Part may be affirmative, and the Characteriſtick negative. Logarithms thus expreſſed are called *binomial Logarithms* (from *binomius, Lat.*) and are better adapted for Practice than the negative ones.

21. To find the binomial Logarithm of a proper Fraction.

* From the Logarithm of the Numerator ſubtract that of the Denominator; remembering, that when we come to the Characteriſticks, the Characteriſtick of the Numerator is to be taken from that of the Denominator; (after having added the one that was borrowed, if any, to the Characteriſtick of the Denominator.)

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* That the binomial Logarithms found by this Rule, are, in Effect, the ſame as the negative Logarithms afore deſcribed, may be thus ſhewn. Let $\frac{n}{d}$ be any proper Fraction; then, putting c for the integral, and r for the fractional Part, of the Logarithm of n; we have $c+r=$ Logarithm of n; and putting t for the Characteriſtick, and $a=$ the decimal Part of the Logarithm of d, then $t+a=$ the Logarithm of d. Hence, by Art. 17, $\overline{t+a}-\overline{c+r}=t+a-c-r$ taken negatively, (that is, changing the Signs,) viz. $-t-a+c+r=$ the Logarithm of $\frac{n}{d}$.

But for the binomial Logarithm we have two Caſes. 1ſt, When r is greater than a, then $t-c$ taken negatively is $-t+c$, ∴ the binomial Logarithm, according to this Rule, is $-t+c+r-a$, which may ſtand thus, $-t-a+c+r$, which is the ſame as the above Expreſſion for the negative Logarithm. 2dly, When r is leſs than a. Firſt, $c+r=\overline{c-1}+\overline{1+r}=$ the Log. of n; now ſubtracting a from $1+r$, there remains $1+r-a$; and taking $c-1$ from t, (according to this Rule,) or, which is the ſame Thing, c from $t+1$, there remains $t+1-c$, which taken negatively is $-t-1+c$ for the Characteriſtick; ∴ $-t-1+c+$ (the decimal Part of the Logarithm) $1+r-a=$ the binomial Logarithm $=-t+c+r-a=-t-a+c+r$, the ſame Expreſſion as that above, for the negative Logarithm. *Q. E. D.*

Example. The binomial Logarithm of $\frac{7.851}{10}$ is $= -1 + .894925$.

For the Logarithm of $7.851 = 0.894925$ by Art. 19.

The Logarithm of $10 = \underline{1.000000}$

The Log. of $\frac{7.851}{10}$ by this Rule $= -1 + .894925$.

But this Logarithm is more commodiouſly wrote thus, $\bar{1}.8942925$, the Daſh over the Characteriſtick ſhewing it is a negative Quantity, and the remaining Part affirmative.

Hence the Log. of $\frac{7.851}{10} = .7851$ is $= \bar{1}.894925$.

of $\frac{.7851}{10} = .07851$ is $= \bar{2}.894925$.

of $\frac{.07851}{10} = .007851$ is $= \bar{3}.894925$.

&c.

Hence it is evident, that the decimal Part of the binomial Logarithm of a decimal Fraction, is the ſame as that of an integral Number of the ſame Figures; and it is alſo manifeſt, that if the ſignificant Figure of the decimal Number is in the firſt Place, the Characteriſtick is -1; if the ſignificant Figure be in the ſecond Place, the integral Part of the Logarithm is -2, &c. And this is the Kind (of Logarithms) we ſhall make Uſe of in the remaining Part of this ESSAY, to expreſs fractional Quantities.

22. In taking out of a Table the Logarithm of any Integer not exceeding 10000, we have the fractional Part by Inſpection; to which we prefix for a Characteriſtick 3, if the Number conſiſts of 4 Figures; 2, if of 3 Figures, &c. that is, the Characteriſtick muſt be one leſs than the Number of Places: But if the Number, whoſe Logarithm is required, be above 10000; then, find the Logarithm of the two neareſt Numbers to it that can be found

in the Table, and ſay, As their Difference : the Difference of their Logarithms :: the Difference between the neareſt Number and that whoſe Logarithm is required : the Difference of their Logarithms nearly; which being either added to, or ſubtracted from, the neareſt Logarithm, according as it is greater or leſs than the required one, will give nearly the Logarithm required. Take an Example.

Let it be required to find the Logarithm of 367182.

The fractional Part of the Logarithm of 3671 is by the Table .564784, and of 3672 is .564903, ∴

The Log. of {367100 is 5.564784 Neareſt N°. 367200
{367200 is 5.564903 The N°. 367182

Their Dif. 100 & of Log. .00119 Diff. 18

∴ As 100 : .00119 :: 18 : .00021 nearly; ∴ 5.564903 — .00021 = 5.564882 nearly = the Logarithm of 367182.

This Method being grounded on Suppoſition, that the Logarithms of Numbers between 367100 and 367200 increaſe or decreaſe equally, according to their Diſtance from 367100 or 367200, is not ſtrictly true, but nearly ſo; and the greater any Numbers are with Reſpect to their Difference, the nearer will thoſe Differences be proportional. And therefore, though this will not give the exact Logarithm, yet it will be nearly ſo, and ſufficiently near the Truth for moſt Purpoſes either of Buſineſs or Pleaſure.

If the Number does conſiſt of both Integers and Fractions, find the fractional Part of the Logarithm as if all its Figures were Integers, and to that fractional Part prefix the Index belonging to the integral Part; *viz.* make the Characteriſtick one leſs than the Number of integral Figures, and you will have the Logarithm ſought.

Laſtly, if the Number whoſe Logarithm is required, be entirely fractional; find the fractional Part of the Logarithm, as if it was a whole Number, and to it prefix the Characteriſtick —1, if the firſt Figure to the left Hand be *Tenths*; but, if the firſt Figure be in the ſecond Place, or Place of *Hundreds*, the Characteriſtick muſt be —2, *&c.*

23. When two or more Logarithms are to be added together, they may be added by the Rules for adding affirmative and negative Quantities in Algebra. That is, 1. If they are all *affirmative*, add them as in common Addition, and the Sum will be the affirmative Logarithm required.

24. *Caſe* 2. If the Characteriſticks of the Logarithms to be added are all *negative*, add the decimal Parts together by common Addition; and if there is any Thing to be carried from the decimal Part to the integral Parts, it is *affirmative*, (by the Nature of binomial Logarithms,) which reſerve: Then find the *Sum* of the negative Characteriſticks; which, being negative, the Difference betwixt this *Sum*, and the Carriage (above reſerved,) with the Sign of the greater, will be the Characteriſtick required.

25. *Caſe* 3. When the Characteriſticks of the Logarithms to be added, are ſome *affirmative* others *negative*, — Add the fractional Parts as uſual, only the Carriage from the decimal Part to the integral being affirmative, muſt be added to affirmative Characteriſticks; whoſe Sum, including the *Carriage*, being found, and alſo the Sum of the negative Characteriſticks, find the Difference of theſe Sums, and to it prefix the Sign of the greater, for the required Characteriſtick.

26. When one Logarithm is to be ſubtracted from another, the Work may be done as in Subtraction of Algebra; by having due Regard to the affirmative and negative Quantities. But to be more particular.

Caſe 1. To ſubtract a leſs affirmative, from a greater affirmative Logarithm; find their Difference, as in common Arithmetick.

27. *Caſe* 2. When it is required to ſubtract a greater affirmative, from a leſs affirmative Logarithm; it is the ſame as to find the Logarithm of a

proper Fraction, and therefore may be found as directed in Article 21.

28. *Caſe* 3. When a *binomial* Logarithm is to be ſubtracted from an affirmative Logarithm, if the decimal Part of the affirmative Logarithm is greater than that of the Binomial, ſubtract the decimal Part of the *Binomial* from the decimal Part of the Affirmative; and prefix the Sum of their Characteriſticks, with the affirmative Sign. But if the *fractional* Part of the binomial Logarithm be greater than that of the Affirmative, add, or ſuppoſe in your Mind 1 to to be added to the fractional Part of the Affirmative; then ſubtract the fractional Part of tha Binomial from that Sum; (but, becauſe we added 1 for Conveniency, we muſt alſo to continue the Equality) add + 1 to the binomial Characteriſtick; but the Characteriſtick of the Binomial Logarithm being negative, the adding + 1 to it is diminiſhing the negative Characteriſtick by Unity; which Characteriſtick ſo diminiſhed, being added to the affirmative Characteriſtick, will give the affirmative Characteriſtick of the required Logarithm.

29. *Caſe* 4. When an *affirmative* Logarithm is to be ſubtracted from a *Binomial*; if the fractional Part of the Binomial be greater than that of the affirmative Logarithm; to the Difference of their fractional Parts, place for a Characteriſtick the Sum of their Characteriſticks with the negative Sign: But if the *decimal* Part of the Binomial is leſs than that of the affirmative Logarithm, add 1 (or ſuppoſe it to be added in your Mind) to the fractional Part of the Binomial; and then ſubtract the decimal Part of the Affirmative from that Sum, and (to continue the Equality) add 1 to the Characteriſtick of the affirmative Logarithm; then the Sum of this, and the Characteriſtick of the Binomial, will, with the negative Sign, be the Characteriſtick of the required Logarithm.

30. *Caſe* 5. When a *binomial* Logarithm is to be ſubtracted from a *Binomial*; if the *fractional* Part of

that to be ſubtracted is leſs than that from which it is to be taken, find the Difference of their fractional Parts, to which prefix the Difference of their Characteriſticks, with the affirmative Sign, if the Characteriſtick of the Logarithm to be ſubtracted is the *greater*; otherwiſe it muſt have a negative Sign. But if the *fractional* Part of the Logarithm to be ſubtracted, be greater than the *fractional* Part of the Logarithm from which it is to be taken, add 1 to the fractional Part of this laſt, that Subtraction may be made; and then, (to continue the Equality, add 1 to the Characteriſtick of the *Minorand*, which Characteriſtick being negative, is done by Subtraction, *viz.* ſubtract 1 from the Characteriſtick of the *Minorand*; then the Difference between the Characteriſtick of the Minorand ſo diminiſhed, and the Characteriſtick of the Subducend, will give the required Characteriſtick; which muſt be *affirmative*, if the Characteriſtick of the *Minorand*, after being diminiſhed as above, is greater than that of the *Subducend*: Otherwiſe *negative*.

31. Before we proceed to the Application of Logarithms, it will be proper to ſhew how to find the *natural* Number anſwering to any given Logarithm, by Help of a Table of Logarithms. And as this is only the Reverſe of Article 22, it will be ſufficient, to hint, 1ſt, That the Index, or Characteriſtick, of the Logarithm, if *affirmative*, will ſhew how many integral Places are in the required Number: And if *negative*, in what Place of Decimals the firſt, or ſignificant Figure, ſtands. Hence, if the Logarithm can be found in the Table, the anſwering Number is found by a bare Inſpection: But if it cannot be exactly found, find the next greater, and next leſſer, and ſay, As the Difference of theſe two Logarithms : the Difference of the anſwering Numbers :: the Difference between the given Logarithm and the neareſt tabular Logarithm : a fourth Number; which added to, or ſubtracted from, the natural Number anſwering to the neareſt tabular Logarithm,

(according as that Logarithm was leſs or greater than the given one,) will give the required Number very near the Truth.

Example. Let it be required to find the natural Number anſwering to the Logarithm 5.564882.

This in the Tables gives the next leſſer and

greater Logarithms,	5.564784 5.564903	The anſwering Numbers	367100 367200
Their Diff.	.000119		100

And 5.564903 — 5.564882 = .000021. Hence, As .000119 : 100 :: .000021; or, which is the ſame in Effect, As 119 : 100 :: 21 : 17.7 nearly. Hence, 367200 — 17.7 = 367182.3, nearly = the required Number.

CHAP. II.

MUTTIPLICATION *by* LOGARITHMS.

32. WE now come to ſhew the Application of Logarithms, having explained their Theory at large; and the firſt Thing is to perform Multiplication by them; which is done by adding the Logarithms of the Factors together, as directed in Art. 23, 24, and 25.

33. *Example* 1. Multiply 14 by 4.

Log. of	14 is	1.1461280
	4 is	0.6020600
Their Sum		1.7481880

anſwers in the Table to 56, the required Product.

34. *Example* 2. Multiply 203 by 0.25.
Log. of 203 is 2.3074960

Log. of 0.25 is $\bar{1}.3979400$

This Log. 1.7054360 anſwers in the Table to 50.75.

35. *Example* 3. Multiply 7.677 by $\frac{5}{8}$.
To Log. of 7.677 = 0.8851915
Add Log. of 5 = 0.6989700

From 1.5841615
Subtract Log. of 8 = 0.9030900

Gives Log. of 4.798 nearly, 0.6810715

Or thus: Put $\frac{5}{8}$ into Decimals, *viz.* $\frac{5}{8}$ = .625.
Then, Log. 7.677 = 0.8851915

Log. of $\frac{5}{8}$ or of .625 = $\bar{1}.7958800$

Sum is 0.6810715 as above.

36. *Example* 4. Multiply 0.55 by .023.

Log. .55 is $\bar{1}.7403627$

Log. .023 is $\bar{2}.3617278$

Log. of .01265 $\bar{2}.1020905$

37. *Example* 5. Multiply 80 by .003.
Log. 80 = 1.9030894

Log. .003 = $\bar{3}.4771212$

Log. .24 = $\bar{1}.3802116$

47. *Example* 8. Divide .01265 by .55.

Log. of .01265 = $\bar{2}.1020905$

Log. of .55 = $\bar{1}.7403627$

Log. of .023 = $\bar{2}.3617278$

CHAP. IV.

Of the APPLICATION *of* LOGARITHMS *to* CIRCULATING DECIMALS.

48. AS in ſome Calculations, in which we propoſe to be very accurate, we may have Occaſion to find the Logarithms of *circulating Decimals*; we ſhall explain the Method by which they may be found, as follows.

The RULE. Firſt turn the *circulating* Decimal into its equivalent vulgar Fraction, (as ſhewn in Theorems 4, 5, 6, and 7, in Articles 13, 14, 15, 16, 17, 18, 19, 20, and 21, of the *Eſſay on Circulating Decimals*,) and then find the Logarithm of that vulgar Fraction, by Art. 21 of this ESSAY.

49. *Example* 1. What is the Logarithm of $.\dot{4}$?

By Art. 14. of the *Eſſay on Circulating Decimals*, $.\dot{4} = \frac{4}{9}$.

Hence from Log. of 4 = 0.6020600
Subtract Log. of 9 = 0.9542425

Gives Log. of $\frac{4}{9}$ or of $.\dot{4}$ = $\bar{1}.6478175$

50. *Example* 2. What is the Logarithm of $\dot{3}.\dot{5}$?

Solution. By Art. 16 of the *Eſſay on Circulating Decimals*, $\dot{3}.\dot{5} = \frac{350}{99}$.

From Log. of 350 = 2.5440680
Subtract Log. of 99 = 1.9956352

Gives Log. of $\frac{350}{99}$ = 0.5484328

51. *Example* 3. What is the Logarithm of $.2\dot{3}$?

Solution. By Art. 18 of the *Eſſay on Circulating Decimals*, $.2\dot{3} = \frac{21}{90}$.

From Log. of 21 = 1.3222193
Subtract Log. of 90 = 1.9542425

Gives Log. of $\frac{21}{90}$ = $\bar{1}.3679768$

52. *Example* 4. What is the Log. of $2\dot{6}\dot{3}53.5$?

Solution. By Art. 20 of the *Eſſay on Circulating Decimals*, $2\dot{6}\dot{3}53.5 = \frac{2609000}{99}$.

Therefore, from Log. of 2609000 = 6.4164741
Subtract the Log. of 99 = 1.9956352

The Remainder is the Log. of $\frac{2609000}{99}$ 4.4208389

53. The Logarithms of Circulating Decimals being found by the above Method; Multiplication, and Diviſion, are performed by them, after the ſame Manner as is ſhewn of finite Numbers in Chap. 2. and 3. of this ESSAY.

CHAP.

CHAP. V.

The GOLDEN RULE *by* LOGARITHMS.

54. FROM the Nature of the *Golden Rule*, and Logarithms, we have this Rule. Add the Logarithms of the ſecond and third Terms together; from their Sum ſubtract the Logarithm of the firſt Term; and the Remainder will be the Logarithm of the fourth, or required Term.

55. Suppoſe in the Rule of Three Direct we have this Stating, As 4.1 : 5 :: 65 : a fourth Number.

To the Log. of 5 = 0.6989700
Add the Log. of 65 = 1.8129134
2.5118834
Sub. Log. of 4.1 = 0.6127838
Log. of 79.26 nearly 1.8990996

55. If from the Logarithm of 1 be ſubtracted the Logarithm of any Number, the Remainder is called the *Arithmetical Complement* of that Logarithm. Hence, the *Arithmetical Complement* of the Logarithm of any Number, is equal to the Logarithm of the Quotient of 1 divided by that Number.

56. When it is required to divide one Number by another, it is in Effect the ſame, if we divide 1 by the Diviſor, and multiply that Quotient by the given Dividend. (For $\frac{a}{m} = \frac{1}{m} \times a$.) Which is done by Logarithms, by adding the Logarithm of the Dividend to the *Arithmetical Complement* of the Logarithm of the Diviſor.

57. Hence, the Operation of the Example in Art. 55, may ſtand thus:

From Log. of 1 $= 0.0000000$
Sub. Log. of 4.1 $= 0.6127839$

Gives *Arith. Comp.* $= \bar{1}.3872161$

$\bar{1}.3872161$
0.6989700
1.8129134

Gives Log. as above 1.8990995

CHAP. VI.

INVOLUTION *by* LOGARITHMS.

58. TO raiſe any Number to a given Power by Logarithms, multiply the Logarithm of the Root by the Index of the given Power. Thus, for Inſtance, if we are to ſquare any Number, we muſt multiply its Logarithm by 2; if to cube a Number, by 3, *&c.* and the Product (by Art. 4.) will be the Logarithm of the required Power.

Note, If the Root be a Fraction, its Logarithm will be a *Binomial*; in ſuch Caſe, the Carriage from the decimal Part in multiplying being affirmative, and the Product of the Characteriſtick negative, the Carriage muſt be ſubtracted from that negative Product, for the required negative Characteriſtick.

59. *Example* 1. What is the Square of 15?

Log. of 15 $= 1.1760913$
2

Log. of 225 $= 2.3521826$

60. *Example* 2. Cube .55.

Log. of .55 $= \bar{1}.7403627$
3

Log. of .166375 $= \bar{1}.2210881$

CHAP. VII.

EVOLUTION *by* LOGARITHMS.

61. TO extract the Root of any Number, divide the Logarithm of the Power by its Index, *viz.* To extract the ſquare Root, divide by 2; for the Cube Root, by 3, *&c.* See Art. 5.

Note. In extracting the Root of a fractional Number, if we cannot divide the Characteriſtick exactly by the Index of the Power, it will be proper to increaſe the Characteriſtick by the Addition of ſuch a Number as will make it exactly diviſible by the Index of the Power; and then divide the Characteriſtick ſo increaſed; but (ſince increaſing the negative Characteriſtick by any Number is really diminiſhing the Logarithm by that Number, we muſt, in Order to continue the Equality, add it to the Logarithm again; therefore, the next Place of the Logarithm being *Tenths*, each Unit in the Characteriſtick is equal to Ten in the Place of *Tenths*; therefore,) for every Unit which we added to the Characteriſtick we muſt now add 10's to the Place of *Tenths*; and divide as uſual.

62. *Example* 1. Extract the ſquare Root of 225.

Log. of 225 is = 2.3521825

$\frac{1}{2}$

Log. of 15 = 1.1760912

63. *Example* 2. Extract the Cube Root of .166375.

Log. of .166375 = $\bar{1}$.2210881

$\frac{1}{3}$

Log. of .55 = $\bar{1}$.7403627

Here, to the Characteriſtick $\bar{1}$, to make it diviſible by 3, we add $\bar{2}$, and the Sum is $\bar{3}$; $\frac{1}{3}$ of which is $\bar{1}$, for the Characteriſtick of the Log. of the Root: Then, to continue the Equality, we add 2 to the Place of Tenths, which makes how many Times 3 in 2.2, which gives .7, &c.

CHAP. VIII.

The APPLICATION *of* LOGARITHMS *to* FELLOWSHIP.

64. As in Fellowſhip, the firſt and ſecond Terms continue the ſame in all the Statings: From the Logarithm of the ſecond Term ſubtract the Logarithm of the firſt Term, and call the Remainder the *reſerved Logarithm*: Then, this *reſerved Logarithm* being added to the Logarithm of the third Term of each reſpective Stating, will give the Logarithm of the fourth, or required Term of that Stating. The Reaſon of this is manifeſt from the Nature of Logarithms, and Chap. VIII. of the Second of the *Mathematical Eſſays*.

65. *Example.* Suppoſe 4 Men, A, B, C, and D, trade in Company; A put in 50*l.* B, 16*l.* C, 25*l.* and D, 18*l.* 10*s.* they gained 20*l.* 15*s.* What was each Man's Part?

Here the Statings would be,

£. £. £.
As 109.5 : 20.75 :: 50 : A's Share.
109.5 : 20.75 :: 16 : B's Share.
109.5 : 20.75 :: 25 : C's Share.
109.5 : 20.75 :: 18.5 : D's Share.

The Operation by Logarithms will be as follows.

From Log. of £. 20.75 = 1.3170181
Sub. Log. of 109.5 = 2.0394141

The Reſerved Log. = $\bar{1}$.2776040

Log. 50 = 1.6989700

Reſerved Log. = $\overline{1}$.2776040

Sum = 0.9765740 the Log. of A's Part anſwering in the Table to 9.47485 *l.*

Log. of 16 = 1.2041200

Reſerved Log. = $\overline{1}$.2776040

Log. of B's = 0.4817249 anſwers to £. 3.031952.

Log. 25 = 1.3979400

Reſerved Log. = $\overline{1}$.2776040

Log. of C's Part = 0.6755440 anſwering to 4.737425.

Log. of 18.5 = 1.2671717

Reſerved Log. = $\overline{1}$.2776040

Log. of D's Part = 0.5447757 anſwering to 3.5056945.

CHAP. IX.

SIMPLE INTEREST *by* LOGARITHMS.

66. ADD the Logarithms of the Principal, Time. and Rate, together; the Sum will be the Logarithm of the Intereſt.

67. *Example.* What is the Intereſt of 410*l.* 10*s.* for 1 Year and 40 Days, at 5*l. per Cent. per Annum?*

The Time 1 Yr. 40 Ds. = $\frac{40}{365}$ = $\frac{81}{73}$. Hence,
The Logarithm of the Time is 0.0451622
Log. of 410.5 the Principal = 2.6133132
Log. of .05 the Rate = $\overline{2}$.6989700
Sum = 1.3574454 the Log. of 22.7743 *l.* = 22 *l.* 15 *s.* 5 *d.* $\frac{3.3}{4}$ for the Intereſt.
Q. E. I.

CHAP.

CHAP. X.

COMPOUND INTEREST *by* LOGARITHMS.

68. THE Rule. Multiply the Logarithm of the Amount of 1 *l.* for one Year, by the Time; and to the Product add the Logarithm of the Principal; the Sum will be the Logarithm of the Amount.

69. *Example.* What will 210*l.* 7*s.* 6*d.* amount to in 3 Years, at 5 *per Cent. per Annum*?

Solution. Firſt, 210*l.* 7*s.* 6*d.* = 210.375 *l.* and the Amount of 1 *l.* for one Year at 5 *per Cent. per Annum* = 1.05.

Hence multiply the Log. of 1.05 = 0.0211893
By the Time = 3

0.0635679
Add Log. of 210.375 = 2.3229940

Log. of Amount 243.5353 *l.* = 2.3865619

Hence the Amount is 243*l.* 10*s.* 8*d.* $\frac{1}{2}$ nearly.

70. *Scholium.* Mr. *Townley*, and a few others, inſtead of the negative Characteriſticks write their arithmetical Complements: Thus, for the Log. of .05, they would write $\overline{8}$.6989700, and for the Log. of .005 place down $\overline{7}$.6989700, &c. And the Rule for adding ſuch Logarithms is, If your Characteriſticks be negative, add them as in common Arithmetick, only noting, that if the Sum of the Characteriſticks be leſs than 10, add 10; if juſt 10, add Unity; if more than 10, caſt 10 away, the Sum, or Remainder, will be the required Logarithm, according to their Method. But if the Characteriſticks are of different Names, *viz.* one affirmative, the other negative, add them together as in common Addition; but if the Sum be 10, or more than 10, caſt 10 away, the Remainder will be affirmative; but if leſs than 10, negative.

Example. Multiply 203 by 25.

Log. of 203 is 2.3074960

Their Log. of 25 is $\overline{9}$.3979400

Log. of 50.75 = 1.7054360

As for Diviſion by Logarithms, they perform it after this Manner, *viz.* If one or both be negative, and the Characteriſtick of the Dividend is leaſt, add 10 to it; but if the Characteriſtick of the Dividend be the greater, ſubtract as uſual, and the Characteriſtick of the Remainder in the firſt Caſe will be affirmative, in the latter, negative.

Example. Divide .01265 by .55.

Log. of .01265 with their Index $\overline{8}$.1020905

Log. of .55 with their Index $\overline{9}$.7403627

Log. of .023 with their Index $\overline{8}$.3617278

Though this laſt Method, of writing the arithmetical Complements for the negative Characteriſticks, may be ſomething ſooner learnt, by ſuch Perſons as are ignorant of the Nature of affirmative and negative Quantities: Yet we would recommend the former Method as the moſt rational, and worthy the Notice of the more ingenious Reader.

Thus much is ſufficient at preſent concerning *Logarithms.* More expeditious Ways of conſtructing Tables, *&c.* will be ſhewn hereafter.

The END.

AN

ESSAY

ON THE

ELEMENTS

OF

PLANE GEOMETRY.

In which the moſt uſeful Propoſitions of the celebrated Geometricians, both ancient and modern, are clearly demonſtrated; and the Conſtructions and Uſes of the neceſſary Inſtruments explained.

By BENJAMIN DONN.

Ὁ Θεὸς ἀεὶ γεωμέτρει. PLATO.

PREFACE.

GEOMETRY and *Arithmetick*, as *Plato* juſtly called them, are "the two Wings by which the Minds of Men mount up to Heaven;" that is, by which they contemplate and diſcover the Motions and Properties of the cœleſtial Bodies.

It is an Obſervation of *Plutarch*, that "GOD adorneth and layeth out all the Parts of the World according to Quantity, Proportion, and Similitude:" Or, in *Solomon*'s Words, "GOD diſpoſes all Things by Number, Weight, and Meaſure." Hence *Plato*, with Reaſon, ſaid, ὁ Θεὸς ἀεὶ γεωμέτρει, GOD always acts according to Geometry.)

Geometry taken in its greateſt Extent, includes almoſt every Branch of the Mathematicks; in which extenſive View, what we have ſaid of Mathematicks in general, in our Preface to the *Mathematical Eſſays*, is applicable to this. But even taken in the narrow Limits to which it is confined in this Eſſay, it is a neceſſary Introduction to almoſt all the Mathematicks, and the practical Sciences depending thereon; for the Architect, Dialliſt, Gauger, Engineer, &c. could do nothing without Geometry, it being as it were the Spring which ſets them all at Work.

[*] "Beſides the Aids which every Branch of the Mathematics derives from Geometry, the Study of this Science is of infinite Advantage in the Uſes of Life. It is always good to think and reaſon right; and it has been juſtly ſaid, that there is no better practical Logic than Geometry". It will ſerve as the ſureſt Means to give our Reaſon the firſt Habitude and Bent of Truth;" and will "teach us to operate upon Truths, to trace the chain of them, ſubtle and almoſt imperceptible as it frequently is, and to follow them to the utmoſt Extent of which they are capable.

"The *geometrical Spirit* is not ſo much confined to *Geometry*, that it cannot be taken off from it, and transferred to other Branches of Knowledge. Works of moral Philoſophy, Politics, Criticiſm, and even Eloquence, *cæteris paribus*, would have additional Beauties if compoſed by *Geometricians*. It ſo ſtrengthens and enlarges the Mind, that, if we believe *Quintilian*, it helps to make an Orator. The Order, Perſpicuity, Diſtinction, and Exactneſs, which have prevailed in good Books for ſome Time paſt, may very probably derive themſelves from this *geometrical* Spirit, which ſpreads more than ever, and in ſome Sort communicates itſelf from Author to Author, even to thoſe who know nothing of Geometry.

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[*] Supplement to Dr. *Harris*'s Lexicon, under the Article *Geometry*, collected from *Lock* and others.

As to the Hiſtory of this ineſtimable Science, we ſhall for the moſt Part give it in the Words of Mr. *Thomas Lediard.*

This Science is ſaid to have been invented by the *Egyptians*, on Account of the Inundations of the *Nile*. This Reaſon might have induced them to cultivate Geometry with a greater Degree of Care and Attention, for diſtinguiſhing again each Man's Property of Land, after the Overflowings of the *Nile*; but Mr. *Rollin* thinks its Origin is undoubtedly of more ancient Date. If we may credit *Joſephus*, (of which Opinion were alſo *Pliny*, *Diodorus Siculus*, and *Cicero,)* the *Aſſyrians* and *Chaldeans* were the firſt after the Flood who applied themſelves to the Study and Practice of the Mathematics. From the *Aſſyrians* and *Chaldeans* it was transferred into *Egypt* by *Abraham*, who, by the Command of God, left his native Country, and coming thither, found the *Egyptians* to be of a remarkable Diſpoſition for learning mathematical Arts; he therefore communicated to them *Arithmetick*, *Geometry*, and *Aſtronomy*. The *Egyptians*, flouriſhing very much in theſe Sciences, gave Occaſion to *Ariſtotle* to ſay, that the *Egyptian Prieſts* were the firſt *Inventors* of the *Mathematical Arts*, who, by Reaſon of their Employments, had ſufficient Time upon their Hands for making Diſcoveries in thoſe Arts. From hence theſe Sciences were carried to *Greece* by *Thales* of *Miletus*, who travelled into *Egypt* for the Improvement of himſelf in Learning. He is ſaid to have found out the 5th, 15th, and 26th Propoſitions of the firſt

Book of *Euclid*, as alſo the 2d, 3d, 4th, and 5th of the fourth Book. He began to obſerve the Equinoxes and Solſtices, and was the firſt who foretold an Eclipſe of the Sun, and alſo one of the Moon to King *Cyrus*; he is therefore juſtly accounted, and ought to be looked upon, as the firſt Founder of the *Mathematics* in *Greece*. He flouriſhed 584 Years before *Chriſt*. (*Vide Stanley*'s Lives of the Philoſophers.)

Thales was ſucceeded by *Pythagoras*, a Philoſopher of *Samos*, who greatly improved and adorned the Mathematics. He found out the 32d, 44th, 47th, and 48th Propoſitions of the firſt Book of *Euclid*, but is eſpecially celebrated for the Invention of the 32d and 47th, for which (as *Apollodorus* ſays) he ſacrificed a * *Hecatomb* to the Gods. He firſt laid open the Doctrine of Incommenſurables and the five regular Bodies. He was the firſt (that we read of) who opened a School wherein Youth might learn thoſe ſublime Arts and Sciences. *Pythagoras* was ſucceeded by *Anaxagoras* of *Clazomenæ* and *Oenopides* of *Chios*; theſe were mentioned by *Plato*: and *Ariſtotle* affirms that *Anaxagoras* wrote a Treatiſe of *Geometry*, and *Proclus* aſcribes the 12th and 23d Propoſitions of the firſt Book of *Euclid* to *Oenopides*. After theſe came *Briſo*, *Antipho*, and *Hippocrates*,

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* *Hecatomb* (ἑκατόμβη, from ἑκατὸν a hundred, and βοῦς Oxen) was a Sacrifice of an hundred Oxen. *Strabo* ſays this Cuſtom came from the *Lacedemonians*; who having 100 Cities in their Country, each City uſed to Sacrifice a Bullock every Year. Some derive the Word from ἑκατὸν and ποῦς a Foot, and hence conclude that a Hecatomb conſiſted only of 25 four-footed Beaſts. The Month in which this Sacrifice was made is called *Hecatombeon*.

Hippocrates, who are all celebrated by *Ariſtotle*, for attempting the Quadrature of the Circle; but of theſe *Hippocrates* was by far the moſt famous; from a Merchant he became a Philoſopher and Geometrician; he firſt attempted the doubling of the Cube by two mean Proportionals, and (as *Proclus* obſerves) was the firſt who wrote Elements, and digeſted into Order what Diſcoveries had yet been made both by himſelf and others. *Democritus* was admirable in Philoſophy and Mathematics, but his Writings were all deſtroyed, (as ſome Authors think) by *Ariſtotle*'s ambitious Deſire of having no other Writings but his own read. *Plato* (who is reported to have been a Scholar to *Theodorus Cyrenæus*) added great Luſtre to the mathematical Sciences: He vaſtly enlarged Geometry with great and noble Additions; he illuſtrated and ſet off his Books of Philoſophy in a mathematical Way, and encouraged whatſoever was admirable either in Mathematics or Philoſophy. Over the Door of his Academy was this Inſcription in Greek, "Let no one ignorant of Geometry enter here."

Out of *Plato*'s Academy proceeded a Number of Mathematicians; hence proceeded *Leodamus* the *Thracian*, who practiſing the Analyſis received from *Plato*, found out many uſeful Things by the Help of it; *Theætetus* the *Athenian*, who is rendered famous as well by *Plato*'s Encomiums (who inſcribed a Dialogue to his Name,) as by his own Inventions; 'twas he that wrote the *Elements* and the Inſcription of the regular Bodies: *Archytas* alſo wrote Elements, and his doubling the Cube is mentioned by *Eutocius*; he has alſo the Honour of being the firſt who applied Mathematics to Uſe in Life. *Gellius* reports, that by his Contrivance a wooden Pigeon was made to fly; he being followed therein by *Dædalus* and others, yielded Matter to the Poets for Fables. *Neoclides* is famous on Account of his Scholar *Leon*, who wrote Elements of the Mathematics, improved them, and made them more fit for the Uſes in Life. *Eudoxus* was not inferior to *Leon*, for to him we are indebted (as ſome Authors affirm,) for the whole 5th and the principal Propoſitions of the 12th Book of *Euclid*. He firſt framed the aſtronomical Hypotheſis, and following *Archytas*, derived down the Springs of Geometry to Mechanics. *Amyclas* the *Heracleot*, *Menechmus*, his Brother *Dinoſtratus*, and *Helicon* of *Cyzium*, all much improved and augmented Geometry. *Theudius* and *Hermotinus* rendered the Elements more perfect and full. *Xenocrates*, the Maſter of *Ariſtotle*, was a Man famous for the Mathematics: A Perſon ignorant of Geometry being deſirous to be his Auditor, he anſwered him thus, "Go thy Way, for thou wanteſt the Handles of Philoſophy." Theſe were all *Platoniſts*.

Ariſtotle's Books are all full of mathematical Arguments: Two of his Scholars are eſpecially celebrated; *Eudemus*, who compoſed a mathematical Hiſtory, and *Theophraſtus*, who wrote two Books of Numbers, four of Geometry, and one of Indiviſibles. To *Ariſteus*, *Iſidore*, and *Hypſicles* we are indebted for the Book of Solids. Laſtly, *Euclid* of *Alexandria*, gathering together the Inventions of others, diſpoſed them into Order, improved and demonſtrated them more accurately, and left us thoſe Elements which lay a ſolid Foundation for all the Branches of mathematical Knowledge. He taught in the Reign of *Ptolemy Lagus, A. R.* 454. The two laſt Books are attributed to *Hypſicles*, and not to him. He died *A. A. C.* 284. He was followed, about 100 Years after, by *Eratoſthenes* and *Archimedes*: The Name of the former was very famous, but his Writings are loſt. We have many Remains of *Archimedes*, but more are ſuppoſed to be loſt. *Polybius*, *Plutarch*, and others have delivered us his Inventions. From him proceeded that Saying, Δὸς ποῦ στῶ καὶ τὴν γῆν κινήσω, i. e. *Let me have where to ſtand, and I will move the Earth which Way I pleaſe.* *Conon* was cotemporary to him: He was a Geometrician and Aſtronomer, and his Death is lamented by *Archimedes* in his Book of the Quadrature of the Parabola. Theſe were followed by *Apollonius* of *Perga*, who was called the Great Geometrician on Account of his Conics, of which there are now extant ſeven Books. To him are alſo aſcribed the fourteenth and fifteenth Books of *Euclid*, which were contracted by *Hypſicles*. *Hipparchus* wrote twelve, and *Menelaus* ſix Books of Subtenſes in a Circle; there are alſo extant three Books of the latter concerning ſpherical Triangles. *Theodoſius* wrote three Books of ſpherical Triangles, which are pretty well known.

In the Year of *Chriſt* 70 flouriſhed *Claudius Ptolemy*: He was not only famous in *Aſtro-*

nomy, but in *Geometry* alſo; which (beſides other Things written by him) his Books of Subtenſes do eſpecially teſtify. He contracted the ſix Books of *Menelaus* and twelve of *Hipparchus* into five Theorems.

Plutarch's mathematical Problems are ſtill extant; and many are to this Day ſenſible of the learned Commentaries of *Eutocius* on *Archimedes*.

The Inventions of *Philo*, *Diocles*, *Nicomedes*, *Sporus*, and *Heron*, concerning doubling the Cube are recited by him. The Genius of the latter was excellent both for *Mechanics* and *Geometry*; his doubling the Cube is likewiſe recommended by *Pappus* before all others.

The Works of *Cteſibius*, to whom we owe the Invention of Pumps, are celebrated by *Proclus*, *Pliny*, and *Athenæus*. The Name of *Geminus* is not in the loweſt Place among *Mathematicians*: *Nicomachus* was famous for arithmetical, geometrical, and muſical Monuments; and *Serenus* well known for his two Books concerning the Section of a Cylinder.

Proclus, *Pappus*, and *Theon* are alſo celebrated among the Ancients: The learned Commentaries upon *Euclid* by *Proclus* ſhew what a Geometrician he was: The Praiſes of *Theon* alſo are deſervedly great; but I ſhall conclude with *Pappus*, the laſt in Time among the Ancients, who in Reputation ought to be reckoned the firſt: He was of *Alexandria*, the moſt fruitful City of great Men, and lived in the Year of *Chriſt* 400: The two firſt of the ſeven Books of his mathematical Collections are loſt; the other five abound with ſo many various and moſt noble Inventions in many Parts of the Mathematics, that they are eſteemed among the chief Monuments of the Ancients which have come down to us.

To particularize the Works of all who have written on Geometry would require a Volume; we ſhall therefore only give a Liſt of as many Authors as we can recollect; *viz. Contractus* in the Year 1000. *Cauſanus* 1440. *Regiomontanus* 1464. *Wernerus* 1480. *Barbarus* 1490. *Durerus* 1500. *Fineus* 1540. *Billingſly* 1570. *Candalla* 1578. *Keckermannus* 1603. *Clavius* 1604. *Severus* 1630. *Cavallerius* 1635. *Ramus* 1636. *Ghetaldus* 1640. *Torricellius* 1644. *P. Gregory* 1647. *Leotandus* 1654. *Fournier*, *Wallis* 1655. *S. Angelus* 1658. *Carragius* 1659. *Fabry* 1669. *Pardie* 1671. *Guarinus* 1671. *Malmſburienſi* 1673. *Bartholinus* 1674. *Dechales* 1674. *Ceva* 1678. *Barrow* 1678. *Henrion* 1683. *D. Gregory* 1684. *Scotus* 1688. *Sturmius* 1689. *Leybourn* 1690. *Oughtred* 1693. *Hugonus* 1698. *Gottignies* 1699. *Guidonus*, *Vivianus* 1701. *Tacquet* by *Whiſton* 1703. *Harris* 1705. *Ward* 1706. Meſſieurs de *Port-Royal* 1711. *Keil* 1728. *Stone* 1731. *Wolſius* 1732. *Martin* 1739. *Clareaut* 1741. *Le Clerc* 1742. *Simpſon* 1747. *Muller* 1748. *Halfpenny* 1753. *Cowley*, * *Simſon* 1756.

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* It is with Pleaſure we acknowledge our Obligations to this great Reſtorer of *Euclid*, (Profeſſor *Simſon* of *Glaſgow*,) for ſeveral valuable Remarks: And therefore the Theorems, &c. which are taken from *Euclid*, are frequently given in the Words of his correct Tranſlation.

What we have chiefly endeavoured in this ESSAY, has been to give clear Demonſtrations of the moſt valuable Propoſitions of *Euclid* and other celebrated Authors, both ancient and modern; that the Reader might acquire a competent Degree of Knowledge in Geometry, for an Introduction to the numerous Branches of Mathematics and Philoſophy, at a moderate Expence of Time and Application.

It is common indeed for Profeſſors to recommend *Euclid* to their Pupils; but though it muſt be allowed that the Demonſtrations of *Euclid* are in general both elegant and accurate, yet if it be conſidered, that at the Time *Euclid* wrote, it was thought ſufficient to underſtand his Elements to be a celebrated Mathematician; whereas they are now only an Introduction to other Arts: It would not, we think, be right to adviſe young Men to ſtudy *Euclid* thoroughly, as it would take up more Time than they can conveniently ſpare. Of this the learned and Reverend Mr. *Bedwell* was ſo ſenſible, that he ſays, * " In what Time, " think you, may a Man learn all *Euclid*, and " ſo by him be made ſkilful in this Art? By " himſelf I know not whether ever or never: " And with the Help of another, although " very expert, I will not promiſe him that he " ſhall attain to Perfection in many Years. — " *Ramus*, my Author, in reading him, con" feſſeth himſelf often to have been at a Stand; " often to have loſt himſelf; often to have hit

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* In his Preface to his Tranſlation of *Ramus*'s Geometry, printed in the Year 1636.

" on a Rock, when he thought he had " touched Land." —— He ſoon after adviſes his Reader by no Means to refuſe the Aſſiſtance of a Teacher, if he can get one, becauſe if he be not of a great Genius, he cannot otherwiſe attain unto any great Perfection, without much Labour and Loſs of precious Time.

The Method of Demonſtration we have choſen to make uſe of in this ESSAY, is the ſynthetick, that it might be underſtood independent of Algebra, and make the Reader acquainted with the ancient Method of Reaſoning, which hath been always juſtly admired for its Accuracy. As *Euclid's* Demonſtrations (except a few which were corrupted by *Theon* and ſome others,) have ſtood the Teſt of more than 2000 Years, we have endeavoured to follow him as nearly as might be conſiſtent with our Plan; and if we are not miſtaken, the Reader will here find all that is * neceſſary by Way of Elements, and may acquire a competent Degree of Knowledge in much leſs Time, and with greater Eaſe, than could be done by the Elements of that juſtly celebrated Author.

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* However, perhaps ſome Readers may find ſeveral Propoſitions in ſome modern Authors not taken Notice of in this Eſſay, which they may think too valuable to be paſſed by in Silence: If ſo, we would only hint, that being deſirous of having the Elements as ſhort as could be conveniently done, for the Eaſe of the young Student: We muſt beg Leave to refer ſuch Readers to the End of our intended Eſſay on Trigonometry, for a Number of other geometrical Propoſitions, as we intend in that Place to give a Collection of many, by Way of Exerciſe in Geometry, Trigonometry, and Algebra.

It is probable this Work will meet with Cenſures of oppoſite Kinds; ſome who are bigotted to Antiquity may perhaps condemn it, becauſe we have taken the Liberty to alter the Order and Demonſtrations of ſome of the Propoſitions; whilſt others who look upon the Ancients with too little Veneration, may blame us for not having taken greater Liberties: But as we have endeavoured to keep clear of each Extreme, the Cenſures of both theſe Sorts of Perſons will give us no Concern.

Bideford, Auguſt 15,
1759.

B. DONN.

A
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BOOK I.

The INTRODUCTION.

1. GEOMETRY originally according to its Etimology ſignified the Art of meaſuring Land; but now is the Science which conſiders the Extenſion and Magnitude of Bodies in general.

2. Geometry is either Theoretical, or Practical. Theoretical Geometry is the Science of Quantity, Extenſion, or Magnitude, abſtractedly conſidered: And is what we ſhall principally treat of in this Eſſay.

3. Practical Geometry is the Application of Theoretical Geometry to the Conveniences of Life and Commerce, &c. which make up the principal Bulk of the Mathematicks. We ſhall therefore only juſt touch on this Part in this Eſſay; intending to treat of its ſeveral Branches in ſome future Eſſays.

4. Geometry may alſo be divided into *Plane* and *Solid*. Plane Geometry treats of the Properties of

Lines and Superficies; and is what we ſhall confine ourſelves to in this Eſſay.

5. Solid Geometry conſiders the Properties of Bodies, with Regard to their Length, Breadth and Thickneſs. Of which hereafter.

6. *Euclid*'s firſt Definition is,

A *Point* is that which hath no Parts, or which hath no Magnitude.

2.

7. A *Line* is Length without Breadth.

3.

8. The Ends, Bounds, or Extremes of a Line are called *Points.**

4.

9. A *right Line*, or *ſtraight Line*, is that which lies evenly between its Points; or which is the ſhorteſt Diſtance from its extreme Points. And a Line, which lies unevenly between its extreme Points, is called a curve Line.

5.

10. A *Superficies*, or *Surface*, is that which hath only Length and Breadth.

6.

11. A *Surface* is bounded or terminated by Lines, or the Extremes of a Superficies are Lines.

7.

12. A *plane Superficies* is that which lieth evenly between its Bounds; or that in which a right Line joining any two Points taken any where in it will be wholly in that Superficies.

8.

Is uſeleſs, and therefore omitted.

9.

13. A rectilineal Angle is the Inclination of two right Lines to one another, meeting in a Point;

---

* As ſome have objected to the firſt Definition, it may not be improper to remark, that hence it follows, a Geometrical Point is not a Quantity, but the End, Term, or Bound of a Quantity; and therefore indiviſible; and conſequently *Euclid*'s Definition that a Point hath no Parts, or Magnitude, is very juſt.

which Point is called the angular Point or Vertex. Or it is the Opening of two right Lines, which meet in a Point.

Note. When there are two or more Angles at the ſame Point, it is common to expreſs the Angle by three Letters, of which the middle one is at the Vertex, and the other two at the Lines forming the Angle. Thus DBC or CBD, *Plate* 1. *Fig.* 1. ſignifies the Angle at B, formed by the Inclination of the Lines DB and CB. And ABD, or DBA the Angle at B, formed by the Meeting of the Lines AB, DB. But if there is but one Angle at a Point, it is generally expreſſed by the ſingle Letter at that Point.

10.

14. When a right Line, ſtanding on another right Line, makes the Angles on either Side of it equal to each other; each of theſe Angles is called a *right Angle*. And the right Line which ſtands on the other is called a Perpendicular to that other. Plate 1. Fig. 2.

11.

15. An *obtuſe Angle* is that which is greater than a right Angle. Fig. 3.

12.

16. An *acute Angle* is that which is leſs than a right Angle. Fig. 4.

13.

Is uſeleſs and therefore omitted.

14.

17. A *Figure* is that which is incloſed by one or more Boundaries.

15.

18 A *Circle* is a plane Figure, contained by one Line, called the Circumference; to which all right Lines drawn from a certain Point within the Figure, are equal to each other. Which right Lines are called the *Radii*. Fig. 5.

16.

19. And that Point within the Figure from which the equal Lines are drawn, is called the *Center.*

17.

20. A *Diameter* of a Circle is a right Line drawn through the Center, and terminated on both Sides by the Circumference.

21. *Scholium.* At the End of this Definition moſt Copies add, "which divides the Circle into two "equal Parts." But this is not properly a Part of the Definition, but a Corollary following from it; the Truth of which plainly appears by ſuppoſing one of the Parts into which the Circle is divided to be laid on the other; for it is manifeſt in ſuch Caſe they will both coincide, otherwiſe the right Lines drawn from the Center to the Circumference could not be equal; and if they coincide, they muſt be equal, by Axiom 8.

18.

22. A *Semicircle* is a Figure contained by a Diameter, and that Part of the Circumference of a Circle cut off by the Diameter.

19.

23. A *Segment* of a *Circle* is a Figure contained by a right Line and the Circumference cut off by it. (This is not uſed in the Elements.)

20.

24. *Rectilineal Figures* are ſuch as are contained by right Lines.

21.

25. *Trilateral Figures*, or *Triangles*, by three right Lines.

22.

26. *Quadrilateral Figures* are contained by four right Lines.

23.

27. *Multilateral Figures*, or *Polygons*, by more than four right Lines.

24.

28. An *Equilateral Triangle* is that which has three equal Sides. Fig. 6.

25.

29. An *Iſoſceles Triangle* is that which has only two Sides equal. Fig. 7.

26.

30. A *Scalene Triangle* is that which has three unequal Sides. Fig. 8.

27.

31. A *right angled Triangle* is that which has one of its Angles a right Angle. Fig. 9.

28.

32. An *obtuſe angled Triangle* is that which hath an obtuſe Angle. Fig. 8.

29.

33. An *acute angled Triangle* is that which has three acute Angles. Fig. 7.

30.

34. A *Square* is a four ſided Figure, whoſe Sides are all equal, and Angles all right Angles. Fig. 10.

31.

35. An *oblong*, *rectangle*, or *right angled Parallelogram*, is that four ſided Figure which has all its Angles right Angles, but not all its Sides equal.

32.

36. A *Rhombus* is that which hath four equal Sides, but its Angles not right Angles.

33.

37. A *Rhomboides* is that which has its oppoſite Sides equal to one another, but all its Sides are not equal, nor its Angles right ones.

Note. For a Definition of a *Parallelogram*, ſee Art. 97.

38. *Scholium.* In moſt Copies of *Euclid*, there is another Condition added, *viz.* “ that its oppoſite “ Angles are equal.” But this is not neceſſary, for we ſhall demonſtrate hereafter, that all quadrilateral Figures whoſe oppoſite Sides are equal, have alſo their oppoſite Angles equal. (See Art. 98.)

34.

39. All other quadrilateral Figures beſides theſe are called *Trapeziums.*

35.

Fig. 14. 40. If two Lines AB, CD, meet any other Line BD, ſo as to make the Angles *a* and *b*, at the ſame Side equal, they are called *parallel Lines.*

N. B. We have here given a different Definition from *Euclid*, becauſe it will render ſome of our Demonſtrations more compendious, *&c.* However its Agreement with *Euclid*'s, that two parallel Lines if infinitely produced could never meet, is demonſtrated in Article 96.

POSTULATES.

1.

41. Let it be granted that a right Line may be drawn from any one Point to another.

2.

42. That a finite right Line may be produced to any Length in a right Line.

3.

43. And that a Circle may be deſcribed from any Center, at any Diſtance from that Center.

Scholium. And that Diſtance is called the *Radius.*

4.

44. That from a given Point and right Line may be drawn, either parallel to another right Line; or ſo as to make any given Angle with that other right Line.

N. B. In demonſtrating the following Theorems we ſhall not refer to theſe Poſtulates; becauſe the Learner will readily ſee where they are ſuppoſed.

AXIOMS.

AXIOMS.

1.

45. Things equal to one and the ſame Thing, or to equal Things, are equal to each other.

2.

46. If to equal Things, equal Things be added, the Whole will be equal.

3.

47. If from equal Things, equal Things be taken away, the Remainders will be equal.

4.

48. If equal Things be added to unequal Things, the Wholes will be unequal.

5.

49. If equal Things be taken from unequal Things, the Remainders will be unequal.

6.

50. Things which are double to one and the ſame Thing, are equal to one another.

7.

51. Things which are halves of the ſame or equal Things, are equal to one another.

8.

52. Magnitudes which coincide, that is, which exactly fill the ſame Space, are equal to one another.

9.

53. The Whole is greater than its Part.

10.

54. Two right Lines cannot incloſe a Space

11.

55. All right Angles are equal to one another.

12.

56. This ſeems to be improperly placed amongſt the Axioms, and has been the Cauſe of much Debate amongſt both the ancient and modern Geometricians; for which Reaſon it is here omitted, and a more uſeful one inſerted, *viz.*

All the Parts taken together are equal to the Whole. *Note.* This is not in *Euclid*, though that illuſtrious Author frequently makes Uſe of it.

Characters.

57. For Brevity the following Signs are made uſe of.

	Signification.
+	*Plus*, or *more*, or *and*, or to be *added*.
—	*Minus*, or *leſs*, or that the Quantity following is to be ſubtracted, or taken from another.
×	A *Rectangle*, thus AB×BD denotes the Rectangle contained under the Length and Breadth denoted by the Lines AB and BD. But we generally expreſs a Square by putting the Figure 2 over the Letter, or Letters, repreſenting its Side. Thus, AB^2 ſignifies the Square whoſe Side is AB, or which is the ſame the Rectangle AB×AB.
: ::	Is to, or to, So is. When four Quantities A, B, C, D, are proportional, that is, when as A is to B ſo is C to D; we generally write them thus, A : B :: C : D. But when three Quantities A, B, C, are proportional the ſecond is repeated thus, as A : B :: B : C.
=	*Equal* to.
Γ	The Greek Letter Gamma denotes *greater than*.
Λ	The Greek Letter Lambda in a vertical Poſition ſignifies *leſs than*; but,
∠	In an horizontal Poſition ſignifies *Angle*; and with the Addition of the Letter *s*, as ∠'s denotes *Angles*.
‖	In a vertical Poſition ſignifies *Parallel*.
△	Signifies *Triangle*, and with the Addition of the Letter *s*, as △'s, is for Triangles.
R	*Radius*.

Rt.	Right. Thus, Rt. ∠. is for Right Angle.
⊥	Denotes a Perpendicular.

When a Number is prefixed to any Quantity; it ſhews how many Times that Quantity is to be taken or repeated: Thus, 6 A, denotes ſix Times the Quantity ſignified by A.

In the following Propoſitions, for *Right Line*, we ſhall only write, *Line*.

Though we ſhall make Uſe of the algebraic Signs as above explained, yet we ſhall give Demonſtrations purely geometrical; for, to uſe the Words of * Mr. *Simpſon*, (who is juſtly celebrated for his great Skill in mathematical Learning,) "It is not the Uſe of "Symbols, (which ſome more ſcrupulous than diſ-"cerning have condemned,) but the Ideas annexed "to them, that render the Conſiderations *geometri-"cal* or *ungeometrical*. In pure Geometry Regard "is always had to the abſolute Quantity of ſome "one of the three Kinds of Extenſion, abſtractedly "conſidered; and whatever Symbols are uſed here, "are to be conſidered as expreſſive of the Quanti-"ties themſelves, and not as any Meaſures, or nu-"merical Values of them. Thus, by A × B, ta-"ken in a geometrical Senſe, we have an Idea, not "of the Product of two Numbers, (as in the al-"gebraic Notation,) but of a real, rectangular "Space, contained under two Right Lines, repre-"ſented by A and B, and two others equal to them.

"So likewiſe, $\frac{B \times C}{A}$ is not to be underſtood here "in the Light of an algebraic Fraction, but as a "Right Line, which is a fourth Proportional to "three Right Lines repreſented by A, B, and C.— "Theſe Diſtinctions are abſolutely neceſſary to thoſe "who would have an accurate Idea of the Sub-"ject."

---

* In his ſelect Exerciſes.

Pl. I. Fig. 15. 58. THEOREM 1. *If two Triangles,* ABC, DEF, *have two Sides of the one,* AB, AC, *equal to two Sides of the other,* DE, DF, *each to each,* viz. AB *equal to* DE, *and* AC *equal to* DF, *and the Angles contained between thoſe equal Sides equal,* viz. *the Angle* A *equal to the Angle* D; *they ſhall have the other Side* BC *equal to the other Side* EF, *the Triangle* ABC *equal to the Triangle* DEF, *and the other Angles in one Triangle ſhall be equal to their correſponding ones in the other Triangle, each to each,* viz. *thoſe to which the equal Sides are oppoſite; that is, the Angle* B *equal to the Angle* E, *and the Angle* C *equal to the Angle* F.

For if the △ ABC be applied to DEF, ſo that the Point A may coincide with D, and the Line AB with DE, the Point B will coincide with E, becauſe AB = DE; and the Side AC will coincide with DF, becauſe ∠A = ∠D; and AC being = DF, the Point C will fall upon the Point F. Hence the Points, B and C, coinciding reſpectively with the Points E and F, the Baſe BC muſt be = the Baſe EF, otherwiſe two Lines would contain a
[a] 10 Ax. Space, which is [a] abſurd. ∴ the Side BC coincides and is = the Side EF; ∴ the whole △ ABC coincides and is = △ DEF; ∴ ∠B coincides and is = ∠E; and ∠C with and = ∠F. Q. E. D.

Pl.I.F.16. 59. THEOREM 2. *The Angles* B, C, *at the Baſe of an Iſoſceles Triangle* ABC, *and oppoſite to the equal Sides* AC, AB, *are equal to each other; and a Line* AD *biſecting the Angle* BAC, *divides the Baſe into two equal Parts, and is perpendicular thereto.*

For becauſe the Line AD biſects the ∠A, that is, makes the ∠BAD = ∠CAD, and Side AB is = AC, and AD common to both, the △ ABD =
[a] 58. [a] △ACD; ∠B = ∠C, and ∠ADB = [b] ∠ADC;
[b] 14. ∴ AD ⊥ to BC. Q. E. D.

60. Corollary. *Hence every equilateral Triangle is alſo equiangular.*

For the Sides AB, AC, being equal, from above, $\angle B = \angle C$; and if the $\triangle$ be equilateral, the Side $CA = CB$, and $\therefore$ by this Theorem, the oppoſite $\angle$s are equal, *viz.* $\angle A = \angle B$ $\because$ $\angle A = \angle B =$ [b]$\angle C$. Q. E. D. [b] 1 Ax.

61. Scholium. This Theorem is the fifth Propoſition of the firſt Book of *Euclid*, and is by many called the *Aſſes Bridge*, becauſe thoſe who are not capable of learning that Book, generally ſtop at this Place; but as *Euclid*'s Demonſtration is certainly ſomething too difficult to be ſo ſoon entered on, we have taken the Liberty to give a different, and it is ſuppoſed, a much eaſier Demonſtration.

62. Theorem 3. *If two Angles* B, C, *of a Triangle* ABC, *be equal, then the Sides* AB, AC, *ſubtending the equal Angles, will be equal between themſelves.* Pl.1.F17.

For, if AB be not $=$ AC, one of them is Γ the other. Let AB be the greater, and let DB be $=$ AC. Then, becauſe in the $\triangle$s DBC, ACB, we have $DB = AC$, BC common to both, and the contained $\angle$s equal, *viz.* $\angle B = \angle C$, the $\triangle$ DBC will in every Reſpect be $=$[a] $\triangle$ACB, the leſs $=$ the greater, which is abſurd. $\because$ AB is not unequal to AC, $\because$ AB muſt be $=$ AC. Q. E. D. [a] 58.

63. Corollary. *Hence every equiangular* $\triangle$ *is alſo equilateral.*

For, by this Theorem, the $\angle B$ being $= \angle C$, the Side AB will be $=$ AC; and in an equiangular $\triangle$, the $\angle A = \angle C$, $\because$ the oppoſite

Sides, BC, AB, are equal. ∵ $AC = BC = AB$, by Axiom 1ſt. Q. E. D.

Pl.1.F18. 64. THEOREM 4. *A Line* AB *ſtanding on another Line* DC, *making with it Angles,* ABD, ABC, *thoſe Angles taken together are equal to two Right Angles.*

For, let EB be a $\perp$ to DC, then the $\angle$s, EBD,
[a] 14. [b] 56. EBC, are [a] Right Angles. But $\angle ABD =$ [b] $\angle EBD$
$+ \angle EBA$; and $\angle EBA + \angle ABC = \angle EBC$; that is,
the $\angle ABD =$ a Rt. $\angle + \angle EBA$, and $\angle EBA + \angle$
$ABC =$ a Rt. $\angle$.

∵ by adding thoſe equal Things together, *viz.*
the Antecedents to the Antecedents, and the Con-
ſequents to the Conſequents, we have $\angle ABD + \angle EBA$
[c] 46. $+ \angle ABC$ [c] $=$ 2 Rt. $\angle$s $+ \angle EBA$; and by taking
away the $\angle EBA$, common to both Sides of the
[d] 47. Equation, we find $\angle ABD + \angle ABC$ [d] $=$ 2 Rt. $\angle$s.
Q. E. D.

65. COROLLARY. 1. *Hence if one of the $\angle$s be a Rt. one, the other is alſo a Rt. $\angle$; and if one of them,* ABC, *is acute, the other,* ABD, *is obtuſe,* et contrà.

66. COROLL. 2. *If two or more Lines ſtand on the ſame Line at the ſame Point, the Angles which they make, taken together, will be = 2 Rt. $\angle$s*; for all the Parts are equal to the Whole.

67. COROLL. 3. *Two Rt. Lines croſſing each other, make 4 $\angle$s, which taken together are = 4 Rt. $\angle$s.*

68. COROLL. 4. *All the $\angle$s made about a Point are = 4 Rt. $\angle$s, by Coroll. 2.*

Pl.1.F.19. 69. *Theorem 5. If two Right Lines,* AB, CD, *interſect each other, the oppoſite Angles are equal,* viz. $\angle AEC = \angle BED$, and $\angle CEB = \angle AED$.

For $\angle AEC + \angle CEB =$ [a] 2 Rt. $\angle$s, and $\angle CEB$ [a] 64.
$+ \angle BED = 2$ Rt. $\angle$s; ∵ $\angle AEC + \angle CEB =$
[b] $\angle CEB + \angle BED$; from which taking away the [b] 45.
$\angle CEB$, common to both, there remains $\angle AEC =$
[c] $\angle BED$. After the ſame Way it may be ſhewn [c] 47.
that $\angle CEB = \angle AED$. Hence the oppoſite $\angle$s are equal. Q. E. D.

70. Theorem 6. *Any two Sides of a Triangle are together greater than than the third Side.*

For the neareſt Diſtance to any two Points being Pl.1.F.15.
[a] a Rt. Line, $BA + AC$ muſt be Γ BC. Q. E. D. [a] 9.

71. *Scholium.* The *Epicureans*, as *Proclus* obſerves, derided this Theorem, it being manifeſt to the weakeſt Capacity, and ∵ required no Demonſtration: However, as the Number of Axioms ought not to increaſed without a Kind of Neceſſity, *Euclid*, who is ſcrupulouſly exact, thought proper to give a formal Demonſtration of it. —— What we have ſaid above is ſufficient.

72. Theorem 7. *If a Line* EF, *croſſes two parallel Lines*, AB, CD, *the alternate Angles*, AGH, GHD, *are equal to each other; and the two interior Angles*, BGH, GHD, *upon the ſame Side, taken together are equal to two Right Angles.* Pl.1.F.20.

For, the Lines AB, CD, being ||, the $\angle AGH$
[a] $= CHF$; and $\angle GHD =$ [b] $\angle CHF$, ∴ $\angle AGH$ [a] 40. [b] 69.
[c] $= \angle GHD$. (After the ſame Manner it might be [c] 45.
ſhewn that $\angle CHG = \angle HGB$.) Again, it has
been juſt ſhewn that $\angle AGH = \angle GHD$, ∵
$\angle BGH + \angle AGH$ [d] $= BGH + \angle GHD$. But [d] 46.
$\angle BGH + \angle AGH$ [e] $=$ 2 Rt. $\angle$s. ∵ $\angle BGH +$ [e] 64.
$\angle GHD$ [c] $=$ 2 Rt. $\angle$s. Q. E. D.

Pl.1.F.20. 73. Theorem 8. *If a Line* EF, *falling upon two other Lines*, AB, CD, *makes the alternate Angles* AGH, GHD, *equal to each other, the Lines* AB, CD, *will be parallel.*

For the $\angle$ AGH being $=$ $\angle$GHD by the Suppo-
[a] 69. ſition, and $\angle$ AGH [a] $=$ EGB, the $\angle$ EGB [b] $=$
[b] 45. $\angle$GHD; $\because$ AB [c] $\parallel$ CD. Q. E. D.
[c] 40.

Pl.1.F.21. 74. Theorem 9. *If two Lines* AB, CD, *be each of them parallel to another Line* EF, *thoſe Lines* AB, CD, *are parallel to each other.*

For AB being $\parallel$ to EF, the $\angle$AGH [a] $=$ $\angle$EHI;
[a] 40. and the Line CD being $\parallel$ to EF, the $\angle$EHI [a] $=$
[b] 45. $\angle$CIK; $\because$ $\angle$ AGH [b] $=$ $\angle$ CIK; $\therefore$ AB is [a] $\parallel$
CD. Q. E. D.

75. *Scholium.* From the bare Definition of parallel Lines, plainly follows this Corollary; *that if a Line is* $\perp$ *to another Line, it is alſo* $\perp$ *to all the Parallels of that other Line.* — See Art. 96.

Pl.1.F.22. 76. Theorem 10. *If any Side* BC *of a* $\triangle$ ABC *be produced, the external Angle* ACD *will be equal to the two internal and oppoſite Angles* A *and* B.

For let the Line CE be $\parallel$ to BA; then the
[a] 40. $\angle$ECD [a] $=$ $\angle$B; and, as the Line AC croſſes two
[b] 72. $\parallel$ Lines BA, CE, the alternate $\angle$s are [b] equal, *viz.*
[c] 46. $\angle$ ACE $=$ $\angle$ A. $\because$ $\angle$ ECD $+$ $\angle$ ACE [c] $=$
[d] 56. $\angle$ B$+$$\angle$A. But $\angle$ ECD$+$$\angle$ACE [d] $=$ $\angle$ ACD,
[e] 45. $\because$ $\angle$ACD [e] $=$ $\angle$B $+$ $\angle$A. Q. E. D.

77. Corollary. *Hence the external Angle* ACD *is greater than either of the internal and oppoſite Angles* A, *or* B; becauſe it is, by this, equal to both.

78. Theorem ii. *The three (interior) Angles of every Triangle are equal to two Right Angles.* Pl.1.F.24.

For, the $\angle A + \angle B$[a] $= \angle ACD$, and to theſe equals adding $\angle ACB$, we have $\angle A + \angle B + \angle ACB$[b] $= \angle ACD + \angle ACB$: But $\angle ACD + \angle ACB$[c] $= 2$ Rt. $\angle$s; $\because$ $\angle A + \angle B + \angle ACB$[d] $= 2$ Rt. $\angle$s. Q. E. D.

[a] 76.
[b] 46.
[c] 64.
[d] 45.

79. Corollary 1. *The three Angles of any Triangle, taken together, are equal to the three Angles of any other Triangle taken together.*

80. Coroll. 2. *If in one Triangle, two of its Angles, taken together, are equal to two Angles of another Triangle taken together, in ſuch Caſe the remaining Angle in one is equal to the remaining Angle in the other Triangle.*

81. Coroll. 3. *If two Triangles have one Angle of the one equal to one Angle of the other, then the Sum of the two remaining Angles of one Triangle is equal to the Sum of the two remaining Angles of the other.*

82. Coroll. 4. *If in any Triangle one of its Angles is a Right Angle, the Sum of the other two is a Right Angle.*

83. Coroll. 5. *Hence, if in a right-angled Triangle one of the acute Angles is given, the other is alſo known, it being the Difference between the given Angle and a Right Angle.*

84. Coroll. 6. *When the Angle contained between the Sides of an Iſoſceles Triangle is a Right Angle, the other two Angles are each of them half a Right Angle.*

For one of the Angles being a Rt. ∠ by the
[a] 82. Suppoſition, the Sum of the other two is [a] = Rt. ∠.
[b] 59. But the ∠s at the Baſe of an Iſoſceles △ are [b] equal;
∵ each of them muſt be = $\frac{1}{2}$ a Rt. ∠, both being equal to a Rt. ∠, as hath been juſt ſhewn.

[a] 60. 85. COROLL. 7. *An equilateral Triangle being* [a] *equiangular, each Angle muſt be equal to one Third of two Rt. Angles; or, which is equivalent, two Thirds of one Right Angle.*

86. COROLL. 8. *If in any Triangle, one of its Angles is either a right or an obtuſe Angle, the other two will be each of them acute.*

87. COROLL. 9. *All the interior Angles of any rectilineal Figure, together with four Right Angles, are equal to twice as many Right Angles as the Figure has Sides.*

Pl.1.F.23. For any rectilineal Figure, ABCDE, can be divided into as many Triangles as the Figure has Sides, by drawing Lines from any Point F within the Figure, to each of its angular Points. And all the
[a] 78. Angles of theſe Triangles are [a] = twice as many Rt. ∠s as there are Triangles, that is, as the Figure has Sides. And the ſame ∠s are = the ∠s of the Figure, together with the ∠s at the Point F, which is the common Vertex of the Triangles; that is, [b] together with four Rt. ∠s. Therefore this Corollary is true.

[b] 68. 88. COROLL. 10. *All the exterior Angles of any rectilineal Figure are together equal to four Right Angles.*

Pl.1.F.24. Becauſe every interior ∠ABC, with its adjacent
[a] 64. exterior ∠ABD, is [a] = 2 Rt. ∠s; ∵ all the interior together, with all the exterior ∠s of any Figure, are

are = twice as many Rt. ∠s as the Figure has Sides, that is, by the laſt Corollary, they are = all the interior ∠s of the Figure, together with four Rt. ∠s. ∵ all the exterior ∠s are = 4 Rt. ∠s.

89. Theorem 12. *The greater Side of every Triangle is oppoſite to the greater Angle:* That is, in ΔABC, let the Side AC be greater than AB, then, I ſay, the Angle B is greater than the Angle C. Pl.1.F.25.

For AC being Γ AB by the Suppoſition, let D be a Point in the Side AC, ſo that AD be = AB; and ſuppoſe B, D, joined. Then, ∠ADB being the exterior ∠ of Δ BDC, the ∠ADB[a] Γ the interior[a] and oppoſite ∠C: But AD being = AB, the ∠ADB[b] = ∠ABD; ∵ as it has been juſt ſhewn[b] that ∠ADB is Γ ∠C, its equal ∠ABD muſt alſo be Γ ∠C, and much more muſt the whole ∠B (which is Γ∠ABD) be Γ∠C. Q. E. D.

[a] 77.
[b] 59.

90. Corollary. *Hence, in every Triangle, the greater Angle is ſubtended by (that is, is oppoſite to,) the greater Side.*

For by the laſt Article, the greater Side is oppoſite to the greater Angle; conſequently, the greater ∠ is oppoſite to the greater Side. For if any Thing, A, is oppoſite to another Thing, B, it is manifeſt that the Thing B muſt be oppoſite to the Thing A. —— Here again we have taken the Liberty to deviate from *Euclid*'s formal Demonſtration.

91. Theorem 13. *If two Triangles,* ABC, DBC, *have the ſame common Baſe, the two Sides* BD, DC, *of the inſcribed Triangle, taken together, will be leſs than the two Sides* BA, AC, *of the circumſcribing Triangle: But the vertical Angle* BDC *of the inſcribed Triangle will be greater than the vertical Angle* A *of the circumſcribing Triangle.* Pl.1.F.26.

As the Sides BD, DC, are contained within the Sides BA, AC, it is manifeſt the Sum of the circumſcribing Sides BA + AC muſt be Γ thoſe which they circumſcribe, *viz.* Γ BD + DC: For whatever Thing includes another, muſt itſelf be Γ that other. This is ſo plain, that Monſieur *Clairault*, a celebrated *French* Geometrician, cenſures *Euclid* for taking the Pains to demonſtrate it. But if a Demonſtration be required, as the Number of Axioms ought not to be increaſed without a Kind of Neceſſity, it may be done thus. Suppoſe BD produced to meet the Side AC in E. Then in △ ABE we have BA + AE [a] Γ BE, and to each of theſe adding EC, we find that BA + AE + EC is Γ BE + EC, but AE + EC = AC, ∵ BA + AC Γ BE + EC. Again, in △ EDC we know that CE + ED [a] Γ DC, and to each of theſe adding DB, we find that CE + ED + DB Γ DC + DB; but ED + DB = BE ∵ CE + BE Γ DC + DB. But it has been proved that BA + AC Γ CE + BE, much more then is it Γ DC + DB. Q. E. D.

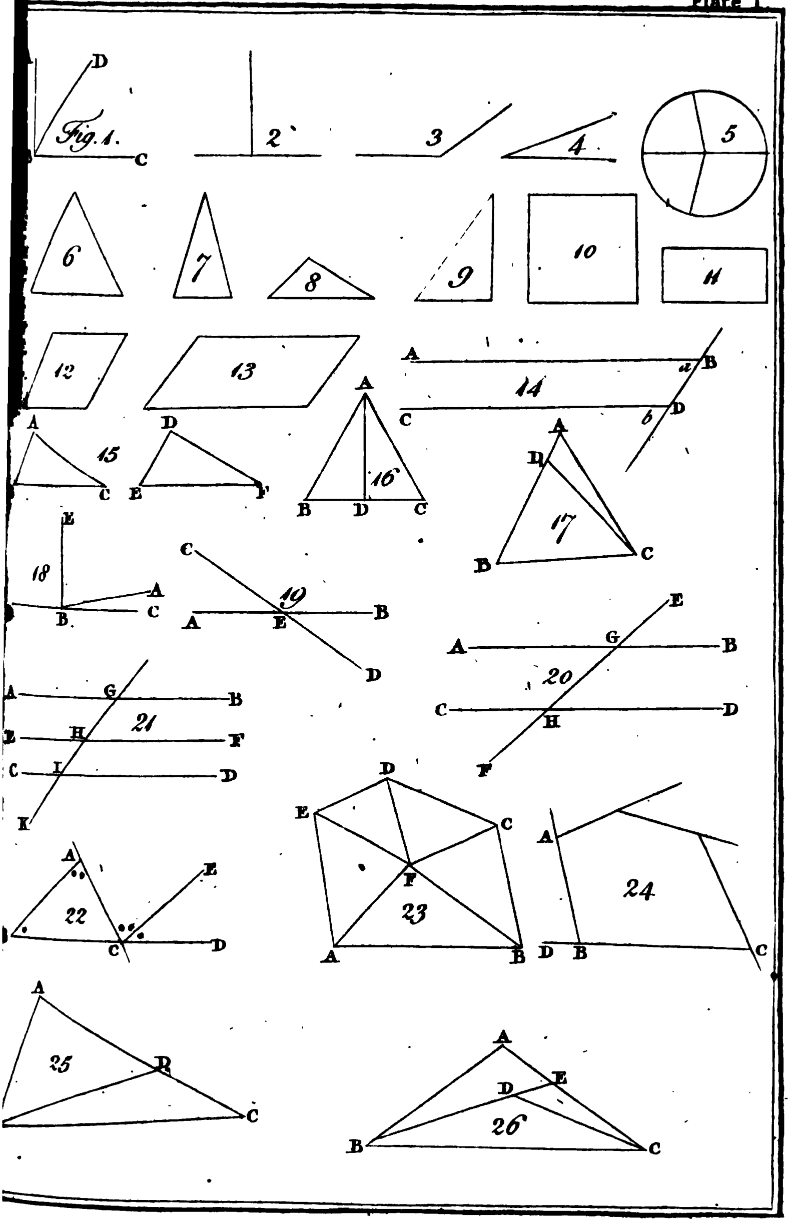
[a] 70.

2. That the ∠ BDC is Γ ∠ A may be thus ſhewn. The exterior ∠ BDC of the △ CDE is [b] Γ the interior and oppoſite ∠ CED; and the exterior ∠ CEB of of the △ ABE is Γ the interior and oppoſite ∠ A. Hence we have proved that the ∠ BDC is Γ ∠ CED, (or, which is the ſame, CEB,) which is Γ ∠ A; much more then muſt ∠ BDC be Γ ∠ A. Q. E. D.

[b] 77.

Pl.2.F.1. 92. THEOREM 14. *If two Triangles* ABC, DEF, *have two Sides of the one equal to two Sides of the other, each to each,* viz. AB = DE, *and* AC = DF; *but the contained Angle of one greater than the contained Angle of the other,* viz. *the Angle* A *greater than the Angle* EDF; *that which has the greateſt Angle between the equal Sides will have the greateſt Baſe,* viz. BC *will be greater than* EF.

Plate I.



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Of the two Sides DE, DF, let DE be that which is not Γ the other; and let DG be a Line = DF, making an ∠ EDG = ∠ A; and let the Points F and G be joined. Then the two Sides AB, AC, of the △ ABC are reſpectively = the two Sides of the △ DEG, and ∠ BAC = ∠ EDG, by the Suppoſition; ∵ the Baſe BC [a] = Baſe EG. And becauſe DG = DF, the ∠ DFG [b] = DGF; but the ∠ DGF Γ ∠ EGF, and ∵ ∠ DFG is Γ ∠ EGF, and much more ſhall the ∠ EFG be Γ ∠ EGF. And the ∠ EFG of the △ EFG, being Γ ∠ EGF, the Side EG is [c] Γ EF: But EG = BC, and ∵ BC muſt be Γ EF. Q. E. D.

[a] 58.
[b] 59.
[c] 89.

93. **Theorem** 15. *If two Triangles* ABC, DEF, *have two Sides of one,* AB, AC, *equal to two Sides of the other,* DE, DF, *each to each,* viz. AB = DE, AC = DF; *but the Baſe of one,* BC, *greater than the Baſe of the other; the Angle contained by the equal Sides of that which has the greater Baſe, ſhall be greater than the Angle contained by the equal Sides of the other;* viz. *the Angle* A *greater than the Angle* D. Pl. 2. F. 2.

This is manifeſt from the preceding Article, being only the Reverſe of that; for if that which has the greateſt ∠ will have the greateſt Baſe, conſequently the greateſt Baſe muſt have the greateſt ∠. However, *Euclid* has thought proper to demonſtrate it in the following Manner, *viz.* If the ∠ A be not Γ ∠ D, it muſt be either = or Λ ∠ D: But it is not equal, for then the Baſe BC would be [a] = EF, but it is not; ∵ ∠ A is not = ∠ D. Nor can the ∠ A be Λ ∠ D; for then the Baſe BC would be Λ [b] EF, but it is not; ∵ ∠ A is not Λ ∠ D, and it has been ſhewn not to be = it; ∵ ∠ A muſt be Γ ∠ D. Q. E. D.

[a] 58.
[b] 92.

94. **Theorem** 16. *If two Triangles have two Angles of one equal to two Angles of the other, each to*

each, and one Side of one equal to one Side of the other, the Triangles are equal in all Reſpects.

The two ∠s of one being = two ∠s of the other, [a] 80. each to each, the remaining ∠ of one [a] = the remaining ∠ of the other; and ∵ the three Angles of one = the three ∠s of the other, each to each. Pl. 2. Fig. 3. Let ABC, DEF, be the two △s, in which Side BC = Side EF, ∠B = ∠E, ∠C = ∠F, and ∠A = ∠D. Then if the △DEF be ſuppoſed to be laid on △ABC, ſo that Side EF agrees with its equal BC; then, ſince ∠B = ∠E, the Side ED will coincide with the Side BA, and ∵ the Point D will be ſomewhere in Line BA; and ∠F being = ∠C, Side DF will coincide with CA, and ∵ Point D muſt be ſomewhere in Line CA; and conſequently, as the Point D is both in the Line BA and CA, it muſt be in the Point where they meet, *viz.* in the Point A, and ſo DE = BA, FD = CA, and the △s in every Reſpect = each other. Q. E. D.

Note. Euclid having demonſtrated this Propoſition, by proving the contrary to be abſurd, we imagined this would be more acceptable, as it is both direct and eaſy to be underſtood.

95. THEOREM 17. *If two Triangles have the three Sides of one equal to the three Sides of the other, each to each, the Angles of one will be equal to the Angles of the other, each to each: Or, the Triangles are equal in all Reſpects.*

Pl. 2. F. 3. For AB being = DE, and AC = DF, if the ∠s A, and D, be one F the other, the Baſes BC, EF, [a] 92. will be one [a] F the other; but they are equal by the Suppoſition: ∵ ∠A muſt be equal to ∠D, and ∵ [b] 58. the △s ABC, DEF, are [b] equal in all Reſpects; and conſequently the ∠A = ∠D, ∠B = ∠E, and ∠C = ∠F. Q. E. D. —— This Demonſtration is alſo different from *Euclid*'s, as are ſome others not taken Notice of.

96. THEOREM 18. *Two parallel Lines* AB, CD, *though indefinitely continued, would never meet; or, in other Words, they continue to be at the ſame Diſtance from each other.*

Let GF be a Perpendicular to the Line AB, and Pl.2.F.4.
HE ⊥ to CD, and ſuppoſe the Points F, E, joined;
then the △s EGF, FHE, have FE common, ∠
GEF =[a] ∠EFH, and ∠s G, and H, each[b] Rt. [a] 72.
∠s; and ∵ the △s have 2 ∠s and one Side of one [b] 14.
= 2 ∠s and one Side of the other, each to each;
and ∵ the ſaid △s are[c] equal in all Reſpects, and [c] 94.
conſequently the Side GF = Side EH. Q. E. D.

97. THEOREM 19. *The oppoſite Sides and Angles of* [*] *Parallelograms are equal to each other, and the Diameter (or Diagonal) biſects them, that is, divides them into two equal Parts.*

Let the Parallelogram ABCD be divided into two Pl.2.F.5.
Parts by the Diagonal BD; then, becauſe AB is ∥
DC, the ∠ABD[a] = BDC; and becauſe AD is ∥ BC, [a] 72.
the alternate ∠s are equal, *viz.* ∠ADB = DBC;
∵ the two △s ABD, CBD, have two ∠s ABD,
ADB, in one, = two ∠s BDC, DBC, of the other,
each to each, and the Side BD common to both △s;
∵ △s ABD, CDB are[b] equal in all Reſpects, (and [b] 94.
conſequently the Parallelogram is divided into two
equal Parts;) ∵ AB = DC, and BC = AD; ∠A
= ∠C, and by adding equals, ∠ABD + ∠DBC =
∠ADB + ∠BDC, that is, ∠ABC = ∠ADC; but
it has been juſt ſhewn that ∠A = ∠C, ∵ the oppoſite Sides, &c. as in the Theorem. Q. E. D.

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[*] A *Parallelogram* is a four-ſided Figure, whoſe oppoſite Sides are parallel. And the Diameter, or Diagonal, is the Line joining two of its oppoſite Angles. And the Altitude is the perpendicular Diſtance betwixt the two oppoſite Sides.

98. THEOREM 20. *All quadrilateral Figures, whoſe oppoſite Sides are equal, have their oppoſite Angles equal.*

Pl.2.F.5. Let the four-ſided Figure ABCD be divided into
two Δs by the Diagonal BD; then AB being =
DC, BC = AD, and BD common to both Δs,
the Δs ABD, CDB, have the three Sides of
one = the three Sides of the other, each to each;
and ∴ the Δs are in all Reſpects [a] = each other:
[a] 95. ∴ ∠A = ∠C; ∠ABD = CDB, ∠ADB = ∠
CBD, and conſequently, by adding Equals, ∠ABD
+ ∠CBD = ∠ADB + ∠CDB, that is, ∠ABC =
∠ADC, but ∠A has been juſt ſhewn to be = ∠C;
∴ the oppoſite ∠s are equal. Q. E. D.

99. THEOREM 21. *Parallelograms, or any other Figures between the ſame Parallels, have equal Altitudes.*

This plainly follows from Article 96; for by that Article the Altitude or ⊥ Diſtance between two ∥ Lines, wherever taken, are equal.

100. THEOREM 22. *Rectangles contained under equal Sides are equal.*

Pl.2.F6. For ſuppoſe the Diagonals BD, FH, to be drawn:
Then, becauſe AD = EH, and AB = EF, by the
[a] 35. Suppoſition, and the Angles A and E both [a] right
∠s; in the Δs ABD, EFH, are two Sides, AD,
AB, and ∠A of one = two Sides EH, EF, and ∠
E of the other, each to each, and ∴ the Δs ABD,
[b] 58. EFH, are [b] equal in all Reſpects. In the ſame
Manner it may be ſhewn that the Δs BDC, FGH,
are equal; and conſequently the whole Rectangle
ABCD = the whole Rectangle EFGH. Q. E. D.

101. COROLLARY. *Hence, if two Squares have the Side of one equal to the Side of the other, the Squares are equal to each other.*

102. THEOREM 23. *Parallelograms upon the ſame Baſe and between the ſame Parallels are equal.*

Pl. 1. F. 7. If the Sides AD, DF, of the Parallelograms ABCD, DBCF, be terminated in one and the ſame Point D, it is plain that each of the Parallelograms is [a] double of the $\triangle BDC$, and conſequently they are [b] equal to each other. Q. E. D.

But, if the Sides AD, EF, oppoſite to the Baſe BC of the Parallelograms ABCD, EBCF, be not terminated in the ſame Point, then, becauſe ABCD is a Parallelogram, AD is [c] $= BC$; by the ſame Reaſon $EF = BC$; $\because$ AD [d] $= EF$, and $\because$ by adding for Fig. 8. and ſubtracting for Fig. 9. DE, which is common, we have ($AD + DE$ [e] $= EF + DE$ for Fig. 8. and $AD - ED$ [f] $= EF - ED$ for Fig. 9. that is,) $AE = DF$; AB is alſo [g] $= DC$, and $\angle A = \angle FDC$, from the Nature of [h] parallel Lines; $\because$ $\triangle$s ABE, DCF, have two Sides of one AB, AE, and contained $\angle A =$ two Sides DC, DF, and contained $\angle FDC$ of the other, and $\because$ the ſaid $\triangle$s are [i] equal in all Reſpects; $\because$ if each $\triangle$ be taken from the whole Figure ABCF, there will remain $ABCD$ [k] $= EBCF$. Q. E. D.

103. COROLLARY 1. *Hence Parallelograms having the ſame Baſe and equal Altitudes, are equal.* See Art. 99.

104. COROLLARY 2. *Every Parallelogram is equal to a Rectangle having the ſame Baſe and equal Altitudes.*

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[a] 97.
[b] 50.
[c] 97.
[d] 45.
[e] 46.
[f] 47.
[g] 97.
[h] 40.
[i] 58.
[k] 47.

105.

105. COROLL. 3. *Hence Parallelograms having equal Baſes and equal Altitudes, are equal.*

106. THEOREM 24. *If a Triangle and Parallelogram ſtand upon the ſame Baſe, and are between the ſame Parallels, the Triangle is half of the Parallelogram.*

Pl.2.F.5. For let BDC be the $\triangle$, and let AB be $\parallel$ and $=$
[a] 97. DC, and AD $\parallel$ and $=$ BC, then is ABCD a [a] Parallelogram, on the ſame Baſe and betwixt the ſame Parallels. Now, one Side BD of the $\triangle$BDC is a Diagonal to the Parallelogram ABCD, and $\because$ the $\triangle$ BDC $= \frac{1}{2}$ the Parallelogram ABCD: But the
[b] 102. Parallelogram ABCD is [b] $=$ any other Parallelogram upon the ſame Baſe and between the ſame Parallels,
[c] 45. $\because$ the $\triangle$BDC [c] $= \frac{1}{2}$ of any Parallelogram on the ſame Baſe and between the ſame Parallels. Q. E. D.

107. COROLLARY 1. *Hence, if a Parallelogram and Triangle have equal Baſes, and are between the ſame Parallels, or, which is the ſame* [a] *Thing, have*
[a] 99. *equal Altitudes, the Triangle is Half of the Parallelogram.*

108. COROLL. 2. *Hence, if a Triangle and Rectangle have equal Baſes and Altitudes, the Triangle is Half the Rectangle.*

109. COROLL. 3. *Hence, if two Triangles have equal Baſes and Altitudes, they are equal to each other.*

For the Triangles being the Halves of Parallelograms of equal Baſes and Altitudes, which Parallelograms are equal, it follows, that if the Wholes (*viz.* the Parallelograms) are equal, their Halves (*viz.* the $\triangle$s) muſt be equal alſo.

110. Scholium 1. On theſe two laſt Theorems and their Corollaries are grounded the Methods of meaſuring, or finding the Areas, or ſuperficial Contents, of Parallelograms and Triangles. But in Order to this it muſt be firſt ſhewn how to find the Area of a Rectangle; which may be thus illuſtrated. Let AB Pl. 2. F. 10. be a Line which we ſuppoſe to be moved towards DC, ſo as to continue always ∥ to its firſt Poſition, *viz.* let the Point A be moved in the ⊥ Line AD, and the Point B in the ⊥ Line BC; then it is manifeſt by ſuch Motion it will deſcribe a Rectangle. And if we imagine the Line AB to be divided into any Number of equal Parts, when the Line is gone in Length one ſuch Part towards D, *viz.* got into the Poſition *ab*, it will have deſcribed a Rectangle A*ab*B, containing as many little Squares as AB was divided into Parts; and when it gets into the Poſition *dc*, two ſuch Parts diſtant from its firſt Poſition, it will have deſcribed a Rectangle containing twice as many Squares as AB contains Parts: And ſo when it arrives at DC, it will have deſcribed the whole Rectangle ADCB = as many little Squares (each Side = one Part of AB) as there are Units in the Product of the Number of Parts in AB multiplied by the Number of Parts in BC: And this is called *Meaſuring*, or finding the *Area* of ſuch Figure. Hence appears the Reaſon why the *Product* of any two Numbers is by *Geometricians* called the *Rectangle.*

Scholium 2. Hence may be ſeen the Reaſon why $3 \times 7 = 7 \times 3$, or generally $A \times B = B \times A$; which is demonſtrated algebraically in our *Mathematical Eſſays.*

111. Hence, ſince any Parallelogram is equal to a Rectangle of equal Baſe and Altitude, *to meaſure any Parallelogram we have this Rule. Multiply the Number of Parts into which the Baſe is ſuppoſed to be*

divided, by the Number of like Parts contained in its Altitude, and the Product will be the Area required: Thus, for Example. If the Baſe of a Parallelogram be 6 Feet, and Altitude 2 Feet, then its Area is = 6 × 2 = 12 ſquare Feet.

112. As to *Triangles*, every Triangle being half of a Parallelogram of equal Baſe and Altitude, Nothing further need be ſaid.

113. THEOREM 25. *The Complements of Parallelograms which are about the Diagonal of any Parallelogram, are equal to each other.*

Pl.2.F.11. Let ABCD be a Parallelogram whoſe Diagonal is
AC; and EH, FG, the Parallelograms about AC,
that is, thro which AC paſſes; and BK, KD, the
other Parallelograms, which make up the whole Figure,
and are therefore called *Complements.* Then
will the Complement BK be = the Complement KD.
For, becauſe ABCD is a Parallelogram, and AC
[a] 97. its Diagonal, the ∆ABC [a] = ∆ADC; and be-
cauſe EKHA is a Parallelogram whoſe Diagonal is
AK, the ∆AEK = ∆AHK; and for the ſame
Reaſon ∆KGC = ∆KFC; ∵ if from the ∆ABC
we take away the ∆s AEK and KGC, and from
the ∆ADC take away the ∆s AHK and KFC, the
[b] 47. remaining Quantities muſt be [b] equal, *viz.* the Pa-
rallelogram BK = the Parallelogram KD. Q. E. D.

Pl.2.F.12. 114. THEOREM 26. *The Diagonals* AC, DB, *of any Parallelogram biſect each other.*

For let the Point of their Interſection be denoted
[a] 97. by E. Then, becauſe AD is [a] =, and ∥ BC, and
[b] 72. the alternate ∠s [b] equal, *viz.* ∠ACB = ∠DAC,
and ∠CBD = ADB, the ∆s AED, BEC, have two
∠s DAE, ADE, and one Side AD of one = two
∠s ECB, EBC, and one Side BC of the other,
[c] 94. each to each; ∴ the ∆s AED, BEC, are [c] equal
in

in all Reſpects, and conſequently the Side AE = Side EC, and Side DE = Side BE, that is, the Diagonals divide each other into two equal Parts. Q. E. D.

115. THEOREM 27. *In any right angled Triangle the Square upon the Side, ſubtending the right Angle, is equal to both the Squares upon the Sides containing the right Angle.*

Let ABC be the △. If we ſuppoſe the Line CA Pl. 2. F. 13.
produced to G, ſince ∠ CAB is a Rt. ∠, the ∠ BAG will be ſo [a] too, and ∵ one Side of the Square [a] 65.
upon BA will fall on the Line [b] AG, and ſo with the Line CA will make one Line CG. For the ſame Reaſon one Side of the Square on CA will with BA make one Line BH. Now, ſuppoſe the Squares deſcribed, and the Points F, C; B, K; A, D; A, E; [b] 34.
joined by Lines; alſo AL || BD. Then becauſe the ∠ DBC = ∠ FBA, each being a right [b] one, ∵ adding the common ∠ ABC, we have the whole ∠ DBA [c] = the whole ∠ FBC; and the Side FB [b] = [c] 46.
BA, and BD = DC: Hence the △s FBC, DBA, have two Sides of one, FB, BC, and contained ∠ FBC = two Sides BA, BD, and contained ∠ DBA of the other, each to each, and ∵ △s FBC, DBA are [d] = each other in all Reſpects: But FB, GC, [d] 58.
are || by the Nature of Squares (for the ∠ FBA = GAB, being each a [b] Rt. ∠, and ∵ FB [e] || GC;) [e] 40.
∵ the Square BG and △ FBC are on the ſame Baſe, and between the ſame Parallels, and ∵ the Square BG [f] = twice the △ FBC; and BD, AL, being [f] 106.
Parallels, the Parallelogram BL = twice the △ DBA; but the △s FBC, DBA, have been juſt ſhewn to be = each other, conſequently the Square BG [g] = Parallelogram BL. [g] 45.
After the ſame Manner it may be ſhewn that the Square CH = the Parallelogram CL; ∵ the Square BG + the Square CH [c] = (the Parallelogram BL + the Parallelogram CL, that is =) the Square BE. Q. E. D. [c] 46.

116. Scholium. This is one of the moſt extenſive and valuable Theorems in Geometry. —— For one Example of its Uſe ſee 1ſt. of the *Mathematical Eſſays*, Article 462.——We ſhall give another, and eaſier Demonſtration after we have treated of *ſimilar Triangles*; when it will alſo be ſhewn to be only a particular Caſe of a more general Theorem.

BOOK II.

Definitions.

117. *Def.* 1. Every right-angled Parallelogram is ſaid to be contained by any two of the Lines which contain one of the right Angles.

118. *Def.* 2. In every Parallelogram either of the Parallelograms about the Diameter, together with Pl.2.F.11. the two Complements, is called the *Gnomon*. Thus the Parallelogram EH, together with the Complements BK, KD, is the Gnomon, which is more compendiouſly expreſſed by the Letters BHF, or GED, which are at the oppoſite Angles of the Parallelograms which make the Gnomon.

Pl.2.F.14. 119. Theorem 1. *If a Line* AB *be divided into any two Parts in* C, *the Square of the whole Line* AB, viz. ABED, *is equal to the Squares of the two Parts* AC, CB, *together with twice the Rectangle contained by the Parts* AC, CB.*

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* It may be demonſtrated algebraically thus. Let $r =$ AC, $x =$ CB, then $\underline{r + x = \text{AB}} = \underline{\text{AD}}$, and ∵ the Area of the * 111, Square ABED* $= \overline{r + x} \times \overline{r + x} =$ (by 1ſt. of the *Mathemat. Eſſays*, Art. 455.) $r^2 + 2rx + x^2$, which is the ſame as the Theorem. Q. E. D. —— Hence, if $x = r$, by writing r for $x = rr + 2rr + rr = 4rr$, the Corollary.

Let CF be $\parallel$ AD or BE, and let AH be = CB; and let HK be a $\parallel$ to AB or DE. Then, becauſe AB[a] = AD, by taking away equals CB, AH, there remains AC[b] = DH. Becauſe HG, DF are Parallels, alſo HD $\parallel$ GF, the oppoſite ∠s of the Figure HGFD are[c] equal to each other; but the ∠D is a Rt. ∠ ∵ the other 3 are Rt. ∠s, and ſo HGFD is a[a] Square each Side = AC. By the ſame Reaſon CBKG is a Square each Side = CB; and by the like Method of Reaſoning ACGH and KEFG are Rectangles whoſe Sides are = AC and CB. But the Squares DG, GB, and two Rectangles AG, GE, make up the whole Square ABED. Q. E. D.

[a] 34. [b] 47. [c] 98. [a] 34.

120. Corollary 1. *Hence the Square of any Line is equal to four Times the Square of half that Line.*

121. Theorem 2. *A Rectangle made by the Sum and Difference of two right Lines is equal to the Difference of the Squares of the ſaid Lines.**

Let AB, AC, repreſent the two Lines, and AK, AG, their Squares. Let HG be produced to HF, till GF = AB, or, which is the ſame, till HF = AC + AB. Let FE be $\parallel$ to AH, and let IK be produced to meet FE in E. Then, becauſe HF = AC + AB, the Sum of the Lines, and IK = AH — AI[a] = AC — AB = the Difference of the Lines, 'tis plain the Rectangle HE is contained under the Sum HF, and Difference IK, of the Lines AC, AB,

Pl. 2. Fig. 15. [a] 47.

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* This Theorem may be eaſily demonſtrated algebraically. Let the greater Line be denoted by a, the leſſer by m, then $a+m$ = their Sum, and $a-m$ = their Difference; then their Rectangle is per Art. 111 = $\overline{a+m}\times\overline{a-m}$ = by the Operation in the Margin, a^2-m^2. the ſame as affirmed in the Theorem. Q. E. D.

$$\begin{array}{r} a+m \\ a-m \\ \hline a^2+am \\ -am+m^2 \\ \hline a^2 \quad - \quad m^2 \end{array}$$

AB, which we are now to ſhew to be = the Gnomon
HDB, the Difference of the Squares AK, AG;
which may be eaſily done, for GF being = AB by
[b] 34. the Suppoſition, = [b] BK, and DG (= CG — CD
[a] 47. = [a] AC — AB) = BC, the Rectangles BD, DF,
[c] 100. are contained under equal Sides, and are ∵ [c] equal
to each other; ∵ to each adding the Rectangle HD
[d] 46. common, we have BD + DH = [d] DF + HD,
that is, the Gnomon HD = the Rectangle HE.
Q. E. D.

122. Theſe two Theorems are ſufficient for our preſent Purpoſe; we ſhall therefore paſs on to the third Book. —— We intend hereafter to give algebraic Demonſtrations to all the Theorems of this Book.

BOOK III.

123. *Def.* 1. EQUAL Circles are thoſe whoſe Diameters are equal, or from whoſe Centers the Lines to the Circumferences are equal: Or, in other Words, they are thoſe that have equal Radii.

124. *Scholium.* This is not, ſtrictly ſpeaking, a Definition, but a Theorem, whoſe Truth is manifeſt; for if the Circles are laid on one another ſo that their Centers may coincide, the Circumferences will alſo coincide, as the Radii are equal, and ∵ the Circles will exactly cover; and be ∵ = each other.

125. *Definition* 2. A Line is ſaid to touch a Circle when it meets the Circle in one Point, and being produced does not touch it in any other Point: And ſuch Line is called a Tangent to the Circle.

126. *Definition* 3. Circles are ſaid to touch one another, which meet, but do not cut each other.

127. *Definition* 4. Lines are ſaid to be equally diſtant from the Center of a Circle, when the Perpendiculars drawn to them from the Center are equal.

128. *Def.* 5. But the Line on which the greater Perpendicular falls, is ſaid to be further from the Center.

129. *Def.* 6. A Segment of a Circle is the Figure contained by a Line and the Circumference it cuts off, (ſee Pl. 2. F. 16.) and the Line is called the Chord.

130. *Def.* 7. The Angle in a Segment is that which is contained by two Lines drawn from any Point in the Circumference (or Arch) of the Segment to the Extremities of the Chord Line, or Baſe of the Segment. See Pl. 2. F. 17.

131. *Def.* 8. And an Angle is ſaid to ſtand upon the Circumference which the Lines that contain the Angle intercept between them.

132. *Def.* 9. The Sector of a Circle is the Figure contained by two Lines drawn from the Center, and the Circumference between them. See Pl. 2. Fig. 18.

133. *Def.* 10. Similar Segments are thoſe in which the Angles are equal, or which contain equal Angles.

134. Theorem 1. *If a Line* CD, *drawn through the Center* E *of a Circle, biſect any Chord* AB, *it will cut it at right Angles. And if the Diameter* CD *is at right Angles to the Chord* AB, *it will biſect it.* Pl. 2. F. 19.

For firſt, Let EA, EB, be joined; then, becauſe
[a] 18. AF = FB by the Suppoſition, and AE [a] = EB being Radii, and FE common, the △s AEF, BEF, have three Sides of one = the three Sides of the other,
[b] 95. each to each, and ∵ they are [b] equal in all Reſpects; conſequently the ∠ AFE = ∠ BFE, and ∵ each =
[c] 14. [c] a Rt. ∠. Q. E. D.

Secondly. But if CD is at Rt. ∠s to AB, then,
[d] 59. ſince AE = EB, the ∠ EAF [d] = ∠ EBF; but ∠ AFE = ∠ BFE, each being a Rt. ∠ per Art. 14. Hence, in the △s AFE, BFE, are two ∠s and one Side of one = two ∠s and one Side of the
[e] 94. other, each to each, and ∵ the △s are [e] equal in all Reſpects: Conſequently, their Baſes FA, FB, are equal to each other. Q. E. D.

Pl.2.F.20. * 134. THEOREM 2. *If any Point* F, *which is not the Center, be taken in the Diameter* AD *of the Circle* ABCD, *whoſe Center is* E; *then, of all the Lines* FB, FC, FG, *&c. that can be drawn from* F *to the Circumference, that* FA, *in which the Center is, is the longeſt, and the other Part* FD *of that Diameter* AD *is the leaſt; and of the others, that which is nearer to the Line* FA *is longer than one more diſtant; thus,* FB *greater than* FC, *and* FC *greater than* FG. *And from the ſame Point* F *there can be drawn to the Circumference only two Lines equal to each other, viz. one on one Side and the other on the other Side of the Diameter* AD, *making equal ∠s with it.*

For 1ſt. Let BE, CE, GE, be joined; then be-
[a] 70. cauſe two Sides of any △ are [a] Γ the third, BE +
[b] 18. EF Γ BF; but BE = [b] AE, ∵ AE + EF (= BE + EF, and ∵) Γ BF. Again, BE being [b] = CE, EF, common, and ∠ BEF Γ ∠ CEF, the Baſe
[c] 92. BF of the △ BEF is [c] Γ the Baſe CF of the △ CEF; and for the ſame Reaſon the Baſe CF Γ Baſe GF. Again, becauſe GF + FE is [a] Γ GE, and GE = ED,

ED, GF + FE muſt be Γ ED, and taking away the Part FE common, GF muſt be Γ FD. Hence we have proved that AF Γ BF, BF Γ CF, CF Γ GF, and GF Γ FD: Which were to be demonſtrated. Secondly. Let DFH be an ∠ = GED, then we affirm the Line FH is = FG, and that no other can be equal to it; for that FH = FG may be thus ſhewn. Becauſe GE = [b] EH, EF common, and ∠ GED = DEF by the Suppoſition, the Baſe GF [d] = Baſe FH. Q. E. D. And that no other Line can be = FG, we thus prove. For if poſſible, let FK be a Line = FG; and FH has been juſt ſhewn to be = FG, and ∵ FK would be [e] = FH, that is, the two Lines FK, FH, = each other, though one is more remote from the Line which paſſes through the Center than the other; which is contrary to what has been proved above, and ∵ FK is not = FH, and ∵ cannot be = to FG. Q. E. D.

[b] 18. [d] 58. [e] 45.

135. THEOREM 3. *Let* ABC *be a Circle, and* D *any Point without it, from which Lines are drawn to the Circumference. Then we affirm, that of theſe which fall upon the concave Part of the Circumference* AEFC, *the greateſt ſhall be* DA, *which paſſes thro' the Center* M; *and the nearer to it greater than the more remote,* viz. DE Γ DF, *and* DF Γ DC. *But of thoſe which fall on the convex Part of the Circumference, the leaſt is* DG, *which coincides with the Diameter* AG *produced; and the nearer to it is always leſſer than the more remote,* viz. DK Λ DL, *and* DL Λ DH. *And only two equal Lines can be drawn from the Point* D *to the Circumference, one upon each Side of the leaſt.*

Pl. III. F. 1.

For, 1ſt. Let the Points ME, MF, MC, MK, ML, MH, be joined; then AM [a] = EM, ∵ adding MD common, EM + MD ([b] = AM + MD) = AD; but EM + MD [c] Γ DE, ∵ alſo DA Γ DE. Again, becauſe ∠ EMD Γ ∠ FMD,

[a] 18. [b] 46. [c] 70.

[a] 18. EM [a] = FM, and MD common to both △s,
[d] 58. EMD, FMD, the Baſe DE [d] Γ Baſe DF. By the ſame Method of Reaſoning DF Γ DC. Hence we have ſhewn that DA Γ DE, DE Γ DF, and DF Γ DC. Q. E. D.

[c] 70. Second. Becauſe MK + KD [c] Γ MD, taking away
[a] 18. equals MK, [a] MG, there will remain KD Γ DG. And becauſe the △ MKD is within the △ MLD, the two Sides MK, KD, taken together muſt be Λ the two Sides ML, LD, which include them, (for it is manifeſt a Thing containing muſt be Γ the Thing contained ;) and taking away equals MK, [a] ML, the Remainder KD Λ the Remainder LD. In the ſame Manner DL is Λ DH ; ∴ DG is the leaſt, DK Λ DL, and DL Λ DH. Q. E. D.

Thirdly. Let the ∠DMB = ∠DMK ; then we affirm that the Line DB = DK, and that no other Line DN can be = DK : For, that DB = DK, may be thus demonſtrated. Becauſe MK [a] = MB,
[a] 18. MD common, and ∠DMB = ∠DMK by the Sup-
[d] 58. poſition, the Baſe DB [d] = Baſe DK. Q. E. D.

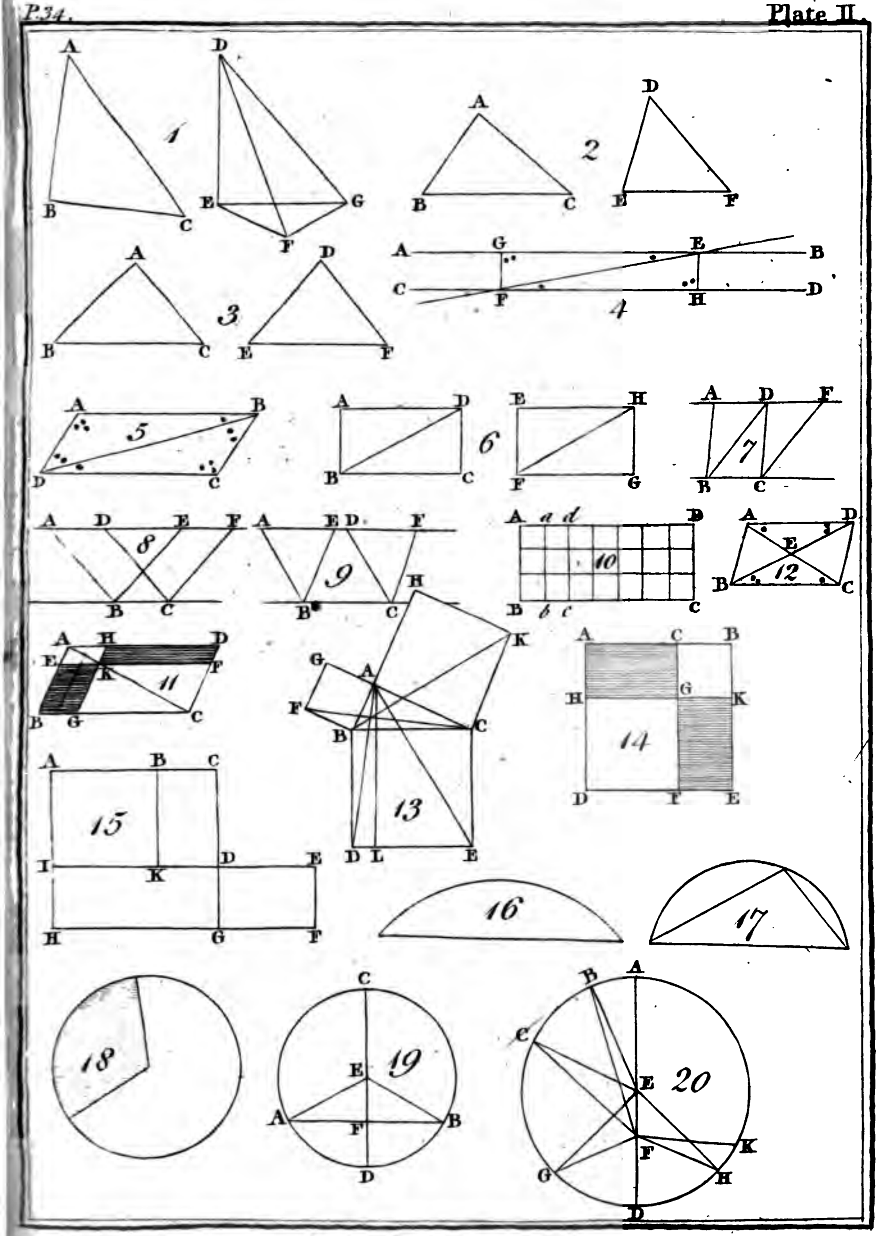
And that no other Line DN can be = DK, may be thus ſhewn : DK is = DB, as has been juſt proved ; ∴ if DN be alſo = DK, then will DN
[e] 45. be [e] = DB, that is, the nearer to the leaſt = the more remote, which is contrary to what has been above demonſtrated, and conſequently impoſſible : ∴ DN is not = DB. Q. E. D.

Pl.3.F.2. 136. THEOREM 4. *If a Point* D *be taken within a Circle, from which there fall more than two equal Lines to the Circumference,* viz. DA, DB, DC, *that Point* D *is the Center of the Circle.*

For if it is poſſible that D is not the Center, let any other Point E be the Center, in the Diameter
[b] 134. FDEG. Then, if D be not the Center, DG is [a] the greateſt Line from it to the Circumference, and DC Γ DB,

P.34.

Plate II.



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r DB, and DB r DA; but they are alſo equal by the Suppoſition, which is impoſſible: ∵ E is not the Center of the Circle ABC, and in the ſame Manner it may be demonſtrated that no other Point but D is the Center. Q. E. D.

137. THEOREM 5. *One Circle cannot cut another in more than two Points.*

For if it be poſſible, let the Circle ABC cut the Circle DEF in more than two Points, *viz.* in B, G, F; and let K be the Center of the Circle ABC, and BF, GK, FK, be joined; then, becauſe within the Circle DEF there is taken the Point K, from which to the Circumference DEF fall more than two equal Lines, KB, KG, KF, the Point K muſt be the [a] Center of the Circle DEF; but it is alſo the Center of the Circle ABC, and ſo the two Circles ABC, DEF, would have the ſame common Center and Radii, and conſequently cannot cut each other, but would coincide, and ∵ properly ſpeaking, would be but one Circle: ∵ one Circle cannot cut another in three Points. Q. E. D. Pl. III. F. 3. [a] 136.

138. A LEMMA, or THEOREM 6. *If two right-angled Triangles have two Sides of one*, AC, CB, *equal to two Sides of the other*, DF, FE, *each to each* (viz. AC = DF, CB = FE;) *the remaining Side of one* AB, *will be* = *the remaining Side of the other* DE. Pl. III. F. 4.

For $AB^2 + BC^2$ [a] $= AC^2$, and $DE^2 + EF^2 = DF^2$: But AC = DF by the Suppoſition, ∵ $AC^2 = DF^2$, ∵ $AB^2 + BC^2$ [b] $= DE^2 + EF^2$; but BC = EF by the Suppoſition, and ſo $BC^2 = EF^2$, ∵ taking away theſe equals, there remains AB^2 [c] $= DE^2$, and conſequently AB = DE. Q. E. D.—— This Demonſtration proves the Theorem to be true, when the Side oppoſite to the Rt. ∠ and one of the other Sides of one, are = the Side oppoſite to the Rt. [a] 115. [b] 45. [c] 47.

∠ and one of the other Sides of the other △. But it is equally true when the two Sides containing the Rt. ∠ in one = the two Sides containing the Rt. ∠ in the other: for it then becomes the ſame as Article 58.

Pl. III. F. 5. *138. THEOREM 7. *Equal Lines*, AB, CD, *in the Circle* ABDC, *are equally diſtant from the Center* E; *and thoſe* AB, CD, *equally diſtant from the Center, are equal to each other.*

Firſt, that if AB = CD, they are equidiſtant from the Center, may be thus demonſtrated. Let EF be ⊥ to AB, EG ⊥ to CD, and join AE and CE. Then the ∠s AFE and CGE being Rt. ∠s,
a 31. the △s AFE, CGE, are [a] Rt. ∠ △s; and AB,
b 134. CD, are [b] biſected by the Lines EF, EG; and ∴ as their Wholes are equal, their Halves muſt be equal alſo, that is, AF = CG. Hence, in the two Rt. ∠ △s AFE, CGE, are two Sides of one AE, AF, reſpectively = two Sides of the other CE, CG, (for AE, CE, are Radii of the ſame Circle and ∴
c 138. equal,) the remaining Side FE of one is [c] = the remaining Side GE of the other; which is all that is meant by ſaying, the Lines AB, CD, are equidiſtant from the Center. Q. E. D.

Secondly. If AB, CD, are equidiſtant from the Center, it may be eaſily demonſtrated that they are = each other. For AE being = CE, and EF = EG, by the Suppoſition, the Rt. ∠ △s AFE, CGE, have two Sides of one AE, EF, = two Sides
c 138. of the other, CE, EG, and ∴ AF [c] = CG; but
b 134. AF, CG, are the [b] Halves of their reſpective Wholes AB, CD; and as the Halves AF, CG, are proved to be equal to each other, their Wholes muſt be ſo too; that is, AB = CD. Q. E. D.

Pl. 3. F. 6. 139. THEOREM 8. *The Diameter* AD *is the greateſt Line in a Circle* ABCD; *and of any others* BC, FG, *that* BC *which is nearer to the Center* E, *is always greater than that more remote* FG.

Let EI, EH, be the Diſtances of BC, FG, from
the Center E; and let BE, CE, FE, GE, be
joined; then, it is manifeſt that AE being [a] = BE, [a] 18.
ED = EC, AD is = BE + CE; but BE + EC [b] Γ [b] 70.
BC, ∵ AD Γ BC. Again, the Δs BEC, FEG,
have two Sides BE, EC, of one = two Sides FE,
GE, of the other; but the contained ∠ of one BEC
Γ the contained ∠ of the other FEC, ∵ BC [c] Γ FG. [c] 92.
Hence we have proved that AB Γ BC, and BC Γ FG;
conſequently Γ any other Line at the ſame Diſtance
from the Center (becauſe Lines at the ſame Diſtance
from the Center are [d] = each other.) Q. E D. [d] 138.

140. *Hence a Line greater than the Diameter, drawn from any Point within the Circle, muſt cut the Circumference; and ∵ any Line within a Circle, being produced both Ways indefinitely, muſt cut the Circumference.*

141. THEOREM 9. *A Line* ED *meeting the Circle* ACG *in the Point* A, *at right Angles to the Radius* BA, *is a Tangent to the Circle at the Point* A. *And no Line* AK *can be drawn from the ſaid Point* A *between the Line* ED *and the Circumference, which does not cut the Circle; or, which is the ſame Thing, no Line* AK *can make ſo great an acute Angle* BAK *with the Diameter* CA, *or ſo ſmall an Angle* FAK *with the Line* AD, *but that it ſhall cut the Circle.* Pl.3.F.7.

Firſt. That ED is a Tangent, may be thus de-
monſtrated. Let any Line BF be drawn; then the
ΔBAF being a Rt. ∠ Δ, BF^2 [a] = $BA^2 + AF^2$, [a] 115.
∵ BF^2 Γ BA^2 ∵ BF Γ BA the Radius, wherever
the Point F is taken in the Line ED, and ∵ the
Point F will fall without the Circle, as BF has
been juſt ſhewn to be Γ the Radius BA, and con-
ſequently Γ its [b] equal the Radius BL. ∵ ED [b] 18.
touches

touches the Circle in no Point but A, and ∵ it is
c 125. a [c] Tangent to the Circle at the Point A. Q. E. D.

2dly. If poſſible, let the Line AK be drawn from A, between the Line ED and the Circumference; and let H be a Point taken in AK, ſo that BH may be at Rt. ∠s to AK, and let G be the Point where the Line GH meets the Circumference. Then, becauſe BHA is a Rt. ∠ Δ, Rt. ∠d at H,
a 82. the ∠BHA [a] = ∠BAH + ∠ABH, the ∠ BHA
b 89. Γ BAH, and ∵ Side BA [b] Γ Side BH; but BA [c] =
c 18. BG, ∵ BG Γ BH, the leſs Γ the greater, which is abſurd; ∵ no Line AK can be drawn from the Point A, between ED and the Circumference, which does not cut the Circle; or, in other Words, " no Line AK can make ſo great an acute ∠BAK with the Diameter CA, or ſo ſmall an Angle FAK with the Line AD, but that the Circumference ſhall paſs between the Tangent AD and Line AK." Q. E. D. And this is all that *Euclid* muſt be underſtood to mean when he ſays, (in the *Greek* Text, and alſo in moſt Tranſlations, that) the ∠ of the Semicircle is Γ any rectilineal ∠, and the remaining ∠ A any rectilineal ∠.

142. COROLLARY I. *Hence if a Line touches a Circle in any Point, a right Line drawn from the Center to that Point will be perpendicular to the touching Line.*

For if that Line be not perpendicular to the Radius at the Point of Contact, it will cut the Circle by this Propoſition.

143. COROLL. 2. *If a Line touches a Circle, and from the Point of Contact a Line be drawn perpendicular to the touching Line, the Center of the Circle ſhall be in that Line.*

144. *Scholium.* From the Manner in which this Theorem is worded in ſeveral Copies of *Euclid*, many ſtrange Paradoxes and Abſurdities have been deduced by ſome modern Mathematicians, which would not have been had they rightly conſidered what *Euclid* muſt mean, and not what the Words ſeem to import.

145. THEOREM 10. *The Angle* BEC *at the Center of a Circle, is double of the Angle* BAC *at the Circumference, upon the ſame Baſe; that is, upon the ſame Part of the Circumference* BC. Pl. III. Fig. 8 & 9.

Becauſe EA[a] = EB, the Angle EAB[b] = ∠ EBA, ∵ ∠ EAB + ∠ EBA = 2∠FAB; but ∠ FEB[c] = ∠ EAB + ∠EBA, ∵ ∠FEB[d] = 2∠ FAB. After the ſame Manner may be ſhewn that the ∠FEC = 2∠FAC; hence, adding in Figure 8th, we have ∠FEB + ∠FEC[e] = 2∠FAB + 2∠ FAC, that is, ∠BEC = 2 ∠BAC. Q. E. D. Or ſubtracting in Figure 9th, we find ∠ FEC − ∠ FEB[f] = 2 ∠ FAC − 2 ∠ FAB, that is, = $\overline{\angle FAC - \angle FAB} \times 2 = 2\,\angle BAC$. Q. E. D.

[a] 18. [b] 59. [c] 76. [d] 45. [e] 46. [f] 47.

146. THEOREM 11. *The Angles in the ſame Segment of a Circle are equal to one another.*

Firſt then, let ABCDE be a Circle, and F the Center; then, if the Segment BAED be F a Semicircle, as in Figure 10th. the ∠s BAD, BED, BFD, ſtand upon the ſame Baſe BD, ∵ the ∠ BAD[a] = $\frac{1}{2}$ ∠BFD, and ∠BED = $\frac{1}{2}$ ∠BFD; ∵ ∠BAD[b] = BED, wherever the Points A, E, are taken in the Circumference BAED. Q. E. D. Pl. III. F. 10.

[a] 145. [b] 45.

Secondly. But if the Segment BAED be Λ a Semicircle, let F be the Center, and ſuppoſe Lines drawn as repreſented in the Figure. Then the Segment Fig. 11.

ment BAEC being Γ a Semicircle, the ∠s in it upon the ſame Baſe BC are by the firſt Caſe equal to each other, that is, ∠ BAC = ∠BEC; and by the ſame Reaſon the ∠s in the Segment CAED on the ſame Baſe CD are = each other, *viz.* ∠CAD = ∠CED; ∵ by adding equals, ∠BAC + ∠CAD
[c] 46. [c] = ∠BEC + ∠CED, that is, ∠BAD = ∠BED. Q. E. D.

Pl. III. Fi. 12. 147. Theorem 12. *The oppoſite Angles of any quadrilateral Figure,* ABCD, *deſcribed in a Circle, are together equal to two right Angles.*

[a] 146. Join AC, BD; then the ∠ CAB [a] = ∠CDB, being in the ſame Segment BADC, and upon the ſame Baſe BC; and the ∠ACB = ∠ADB, being in the ſame Segment ADCB, and upon the ſame Baſe AB. ∵ adding equals, the ∠ADB + ∠
[b] 46. CDB [b] = ∠ ACB + ∠CAB, that is, ∠ADC = ∠ACB + ∠CAB; to each of theſe equals adding the ∠ABC, we have ∠ADC + ∠ABC = ∠ACB + ∠CAB + ∠ABC; but the ∠ACB + ∠CAB +
[c] 78. ∠ABC [c] = 2 Rt. ∠s, ∵ ∠ADC + ∠ABC [d] =
[d] 45. 2 Rt. ∠s. In the ſame Manner it may be ſhewn that the ∠BAD + ∠BCD = 2 Rt. ∠s. Q. E. D.

Pl. III. F. 13. 148. Theorem 13. *The Angle in a Semicircle* BAC *is a right Angle; but the Angle in a Segment* ABC, *greater than a Semicircle, is leſs than a right Angle; and the Angle in a Segment* ADC, *leſs than a Semicircle, is greater than a right Angle.*

Let E be the Center, and Lines be drawn as in
[a] 18. the Figure. Then, AE being [a] = EB, the ∠EAB
[b] 59. [b] = ∠EBA; and EA being = EC, the ∠EAC = ∠ECA. ∵ adding equals, ∠ EAB + ∠ EAC
[c] 46. [c] = ∠ EBA + ∠ ECA, that is, ∠ BAC = ∠
[d] 76. CBA + ∠BCA; but, ∠FAC [d] = ∠CBA + ∠
[e] 45. BCA, ∵ ∠BAC [e] = ∠ FAC, ∵ each of them is

is a [f] right ∠, ∵ the ∠BAC in a Semicircle is a Rt. ∠. Q. E. D. [f] 14.

2dly. The △BAC having been juſt ſhewn to be Rt. ∠d. *viz.* the ∠BAC a Right ∠, the ∠ABC (in the Segment ABC) muſt be [g] Λ a Rt. ∠. Q. E. D. [g] 86.

3dly. The Figure BADC being quadrilateral, the ∠ABC + ∠ADC [h] = 2 Rt. ∠s; but it has been juſt ſhewn that ∠ABC Λ a Rt. ∠, conſequently ∠ADC muſt be Γ a Rt. ∠. Q. E. D. [h] 147.

149. THEOREM 14. *In an oblique Triangle* ACD, *inſcribed in a Circle, the vertical Angle* ADC *exceeds, or is leſs than a right Angle, by the Angle* CAB, *made by the Baſe* AC *and the Diameter* AB, *drawn from either Extremity of the Baſe.* Pl. III. F.14.&15.

Join DB; then, AB being a Diameter, ADCB is [a] a Semicircle, and ∵ ∠ADB [b] = a Rt. ∠, and the ∠CDB [c] = ∠CAB, (being in the ſame Segment CDAB, and ſtanding on the ſame Baſe BC;) ∵ adding equals in Fig. 14. the ∠ADB + ∠CDB [d] = a Rt. ∠ + ∠CAB; but ∠ADB + CDB = the whole ∠ADC, ∵ ∠ADC = a Rt. ∠ + ∠CAB. Q. E. D. And in Fig. 15. ſubtracting equals, the ∠ADB — ∠CDB [e] = a Rt. ∠ — ∠CAB; but ∠ADB — ∠CDB = ∠ADC, ∵ ∠ADC = a Rt. ∠ — ∠CAB. Q. E. D. [a] 22. [b] 148. [c] 146. [d] 46. [e] 47.

150. THEOREM 15. *In equal Circles,* viz. *Circles which have equal Radii, the Angles* A, G, *at the Circumferences, and ſtanding upon equal Chords* BC, EF, *are equal to each other. And if the Angles at the Circumferences are equal, the Chords on which they ſtand will be equal alſo.* Pl. III. F.16&17.

Let D and H be the Centers, and let Lines be drawn as in the Figures. 1. Then, BC being =

EF by the Suppoſition, and BD = CD = EH =
FH being equal Radii, the Δs BDC, EHF, have
three Sides of one reſpectively = three Sides of the
[a] 95. other, and ∵ ∠ BDC [a] = ∠ EHF; but the ∠s
[b] 145. A, and G, are [b] Halves of the ∠s BDC, EHF;
and as the Wholes are equal to each other, 'tis maniſeſt their Halves muſt be ſo too, that is, ∠ BAC = ∠ G. Q. E. D.

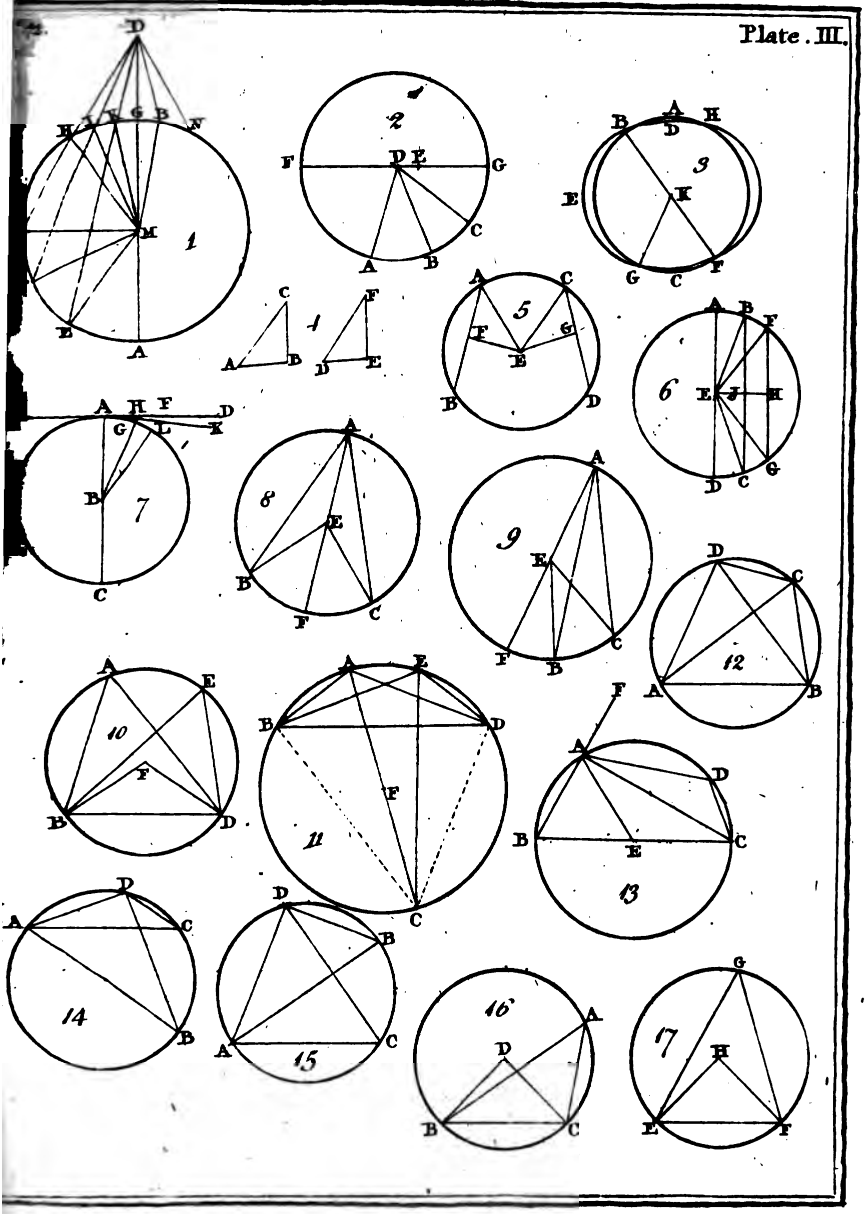
2dly. If the ∠ A = ∠ G; as the ∠s A and G,
[b] 145. are the Halves [b] of the ∠s BDC, CHF, reſpectively,
and as the Halves being equal, the Wholes muſt be ſo too, the ∠ BDC muſt be = ∠ EHF. But BD = EH, DC = HF, being equal Radii, the Δs BDC, EHF, have two Sides BD, CD, and ∠ contained BDC of one = two Sides EH, FH, and ∠ contained EHF of the other; and conſequently the Δs
[a] 95. themſelves [a] equal in all Reſpects; ∵ the Baſe BC =
Baſe EF. Q. E. D.

Pl. IV. Fig. 1. 151. Theorem 16. *Let* AB *be the Diameter of the Circle* ADCB, *and* EF *a Line perpendicular thereto, touching the Circle in the Point* B; *and* BD *a Line cutting the Circle; then I ſay, the Angles made by the touching Line* EF *and cutting Line* BD, *ſhall be equal to the Angles in the alternate Segments*; viz. ∠ DBF = ∠ DAB, *and* ∠ DBE = ∠ DCB, *wherever the Points* A *and* C *are taken in the Segments*, DAB, DCB.

For the vertical ∠ DAB of the Δ DAB is Λ a Rt.
[a] 149. ∠ by the [a] ∠ DBA, made by the Baſe DB and Diameter BA; and ∵ as ABF is a Rt. ∠, muſt be = ∠ DBF. Q. E. D.

Again, the vertical ∠ DCB of the Δ DCB exceeds a Rt. ∠ by the ſaid ∠ DBA, and ∵ as ABE is a Rt. ∠, the ∠ DCB muſt be = (∠ ABE + ∠ ABD) = ∠ DBE. Q. E. D. And the ∠ DAB
[b] 146. continues the [b] ſame wherever the Point A is taken in
the

Plate . III.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the Circumference DAB, as does the ∠DCB whereever the Point C is taken in the Circumference DCB. ∴ this Theorem is true.

BOOK IV.

Of PROPORTION, *containing what is moſt valuable in* Euclid'*s 5th and 6th Books.*

152. *Def.* 1. A Leſſer Magnitude is ſaid to be an *aliquot Part* of a greater Magnitude, when the leſſer meaſures the greater, *i. e.* when the leſſer is contained a certain Number of Times exactly in the greater. Thus, a is an aliquot Part of $3a$, a being contained in $3a$, 3 Times.

153. *Def.* 2. A greater Magnitude is ſaid to be a *Multiple* of a leſſer, when the leſſer meaſures the greater, *i. e.* when the leſſer is contained in the greater, exactly a certain Number of Times. Thus, $3a$ is a Multiple of a, a being contained in $3a$, 3 Times, and $n \times b$ is a Multiple of b, for b is contained n Times in $n \times b$.

154. The 3d *Definition*, " That Ratio is a mutual Relation of two Magnitudes of the ſame Kind to one another in Reſpect of Quantity," is rather metaphyſical than mathematical, and has been the Cauſe of much Controverſy amongſt the modern Mathematicians. Indeed it might well be omitted without any Loſs, and, as Mr. *Simſon* juſtly obſerves, ſeems to be the Addition of ſome unſkilful Editor.

155. *Def.* 4. Theſe Magnitudes are ſaid to have a Ratio to one another, the leſſer of which can be multiplied ſo as to exceed the other.

156. *Def.* 5. The firſt of four Magnitudes is ſaid to have the ſame Ratio to the ſecond which the third has to the fourth, when any Equimultiples whatſoever of the firſt and third being taken, and any Equimultiples whatſoever of the ſecond and fourth: If the Multiple of the firſt be leſs than that of the ſecond, the Multiple of the third is alſo leſs than that of the fourth; or, if the Multiple of the firſt be equal to that of the ſecond, the Multiple of the third is alſo equal to that of the fourth; or, if the Multiple of the firſt be greater than that of the ſecond, the Multiple of the third is alſo greater than that of the fourth.*

157. *Def.* 6. Magnitudes which have the ſame Ratio are called Proportionals.

158. *Def.* 7. Of a greater and leſs Ratio being not uſed in this Eſſay, we have thought proper to omit it, as the Theorems which are obſcurely, (if not inaccurately) demonſtrated by it, may be demonſtrated more clearly by other Means.

159. The ſame may be ſaid of the 8th *Definition*, *viz.* "That Analogy or Proportion is a Similitude of Ratios," as was remarked concerning the 3d. Definition.

160. *Def.* 9. Proportion conſiſts in three Terms at leaſt.

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* This is the true geometrical Definition of the Ancients; however it may not be improper to give alſo the *numerical* or *algebraical Definition*, *viz.* Four Quantities are ſaid to be in direct Proportion to each other, when the Quotient of the firſt by the ſecond is equal to the Quotient of the third by the fourth: Which *Quotients* are called *Ratios*. Or in other Words, *Ratio* is the Number expreſſing how often the Conſequent is contained in the Antecedent; or by which the Conſequent being multiplied, the Product will be equal to the Antecedent.

161. *Def.* 10. When three Magnitudes are Proportionals, the firſt is ſaid to have to the third the duplicate Ratio of that which it has to the ſecond.

162. *Def.* 11. When four Magnitudes are continual Proportionals, the firſt is ſaid to have to the fourth the triplicate Ratio of that which it has to the ſecond, and ſo forward quadruplicate, &c. increaſing the Denomination each Time by Unity, in any Number of Proportionals.

163. *Def.* 12, of *Compound Ratio.* When there are any Number of Magnitudes of the ſame Kind, the firſt is ſaid to have to the laſt of them the Ratio compounded of the Ratio which the firſt has to the ſecond, and of the Ratio which the ſecond has to the third, and of the Ratio which the third has to the fourth, and ſo forward unto the laſt Magnitude.

For Example, if A, B, C, D, be four Magnitudes of the ſame Kind, the firſt A is ſaid to have to the laſt D the Ratio compounded of the Ratio of A to B, and of the Ratio of B to C, and of the Ratio of C to D; or, the Ratio of A to D is ſaid to be compounded of the Ratios of A to B, B to C, and C to D.

And if A has to B the ſame Ratio which E has to F; and B to C the ſame Ratio which G has to H; and C to D the ſame Ratio that K has to L; then, by this Definition, A is ſaid to have to D the Ratio compounded of Ratios which are the ſame with the Ratios of E to F, G to H, and K to L. And the ſame Thing is to be underſtood when it is more briefly expreſſed by ſaying A has to D the Ratio compounded of the Ratios of E to F, G to H, and K to L.

In like Manner, the ſame Things being ſuppoſed, if M has to N the ſame Ratio which A has to D, then, for Shortneſs Sake, M is ſaid to have to N the

Ratio compounded of the Ratios of E to F, G to H, and K to L.

164. *Scholium.* Nothing has been thought more difficult, or cauſed greater Diſputes in Geometry, than the Definition of compound Ratio given by *Theon* in the 5th Def. of the 6th Book of *Euclid*, *viz.* " A Ratio is ſaid to be compounded of Ratios when the Quantities of Ratios being multiplied by one another, makes a certain Ratio ;" which is faulty, for there can be no Multiplication but by a Number, and ſo the Definition will be ungeometrical. As to the Uſe of compound Ratio, it is introduced into Geometry only to render ſome Demonſtrations more conciſe, or that ſome Propoſitions may be more compendiouſly enunciated. It may be further obſerved, " that in any Magnitudes whatever, " of the ſame Kind, A, B, C, D, &c. the Ratio " compounded of the Ratios of the firſt to the ſe- " cond, of the ſecond to the third, and ſo on to " the laſt, is only a Name or Expreſſion, by which " the Ratio which the firſt A has to the laſt D is " ſignified, and by which at the ſame Time the " Ratios of all the Magnitudes, A to B, B to C, " C to D, from the firſt to the laſt, to one another, " whether they be the ſame or be not the ſame, are " indicated ; as in Magnitudes which are continual " Proportionals, A, B, C, D, &c. the duplicate " Ratio of the firſt to the ſecond is only a Name, or " Expreſſion, by which the Ratio of the firſt A to " the third C is ſignified, and by which, at the " ſame Time, is ſhewn that there are two Ratios " of the Magnitudes from the firſt to the laſt, *viz.* " of the firſt A to the ſecond B, and of the ſecond " B to the third or laſt C, which are the ſame to " one another ; and the triplicate Ratio of the firſt " to the ſecond is a Name, or Expreſſion, by which " the Ratio of the firſt A to the fourth D is ſignified, " and by which, at the ſame Time, is ſhewn that " there are three Ratios of the Magnitudes from the

“ firſt to the laſt, *viz.* of the firſt A to the ſecond
“ B, and of B to the third C, and of C to the fourth
“ or laſt D, which are all the ſame to one another;
“ and ſo in the Caſe of any other multiplicate Ra-
“ tios.” And this is the right Meaning of theſe Ratios as uſed by *Euclid*, *Archimedes*, *Apollonius*, and other ancient Geometricians. However, that of *Theon* abovementioned is ſtill retained by moſt Moderns, and indeed may be made Uſe of when we are diſcourſing of Numbers, or of Magnitudes as expreſſed by Numbers, though it would be improper in this on Geometry.

165. *Def.* 13. In Proportionals the antecedent Terms are called homologous to one another, as alſo the Conſequents to one another.

Note. Geometricians make Uſe of the following technical Words to ſignify certain Ways of changing either the Order or Magnitude of Proportionals, ſo as to continue ſtill to be Proportionals.

166. *Def.* 14. *Permutando*, or *alternando*, *by Permutation*, or *alternately:* This Word is uſed when there are four Proportionals, and it is inferred, that the firſt has the ſame Ratio to the third which the ſecond has to the fourth; or that the firſt is unto the third, as the ſecond to the fourth; as is ſhewn in Article 199.

167. *Def.* 15. *Invertendo*, *by Inverſion*; when there are four Proportionals, and it is inferred, that the ſecond is unto the firſt as the fourth to the third. See Art. 198.

168. *Def.* 16. *Componendo*, *by Compoſition*; when there are four Proportionals, and it is inferred, that the firſt, together with the ſecond, is to the ſecond, as the third, together with the fourth, is to the fourth. See Art. 200.

169. *Def.* 17. *Dividendo*, by *Diviſion*; when there are four Proportionals, and it is inferred, that the Exceſs of the firſt above the ſecond is to the ſecond, as the Exceſs of the third above the fourth is to the fourth. See Art. 201.

170. *Def.* 18. *Convertendo*, by *Converſion*; when there are four Proportionals, and it is inferred, that the firſt is to its Exceſs above the ſecond, as the third to its Exceſs above the fourth. See Art. 202.

171. *Def.* 19. *Ex æquali*, (ſcil. *Diſtantia*,) or *ex æquo*, *from Equality of Diſtance*; when there is any Number of Magnitudes more than two, and as many others ſo that they are Proportionals when taken two and two of each Rank, and it is inferred, that the firſt is to the laſt of the firſt Rank of Magnitudes, as the firſt is to the laſt of the others. ——Of this there are the two following Kinds, which ariſe from the different Order in which the Magnitudes are taken two and two.

172. *Def.* 20. *Ex æquali, from Equality*: This Term is uſed ſimply by itſelf, when the firſt Magnitude is to the ſecond of the firſt Rank, as the firſt to the ſecond of the other Rank; and as the ſecond is to the third of the firſt Rank, ſo is the ſecond to the third of the other; and ſo forward in Order, and the Inference is as mentioned in the preceding Definition; whence this is called *Ordinate Proportion.*

173. *Def.* 21. *Ex æquali, in proportione perturbata, ſeu inordinata, from Equality, in perturbate or diſorderly Proportion:* This Term is uſed when the firſt Magnitude is to the ſecond of the firſt Rank, as the laſt but one is to the laſt of the ſecond Rank; and as the ſecond is to the third of the firſt Rank, ſo is the laſt but two to the laſt but one of the ſecond Rank; and as the third is to the fourth of the firſt

Rank, ſo is the third from the laſt to the laſt but two of the ſecond Rank, and ſo forward in a croſs Order; and the Inference is as in the 19th Definition.

174. *Def.* 22. *Similar rectilineal Figures* are thoſe which have their ſeveral Angles equal, each to each, and the Sides about the equal Angles Proportionals.

175. *Def.* 23. Two Magnitudes are ſaid to be reciprocally proportional to two others, when one of the firſt is to one of the other Magnitudes, as the remaining one of the laſt two is to the remaining one of the firſt.

176. *Def.* 24. A Line is ſaid to be cut in *extreme and mean Ratio*, when the Whole is to the greater Segment as the greater Segment is to the leſs.

177. *Axiom* 1. *Equimultiples* of the ſame, or equal Magnitudes, are equal to one another; that is, if A = B, then any Number of Times A = ſame Number of Times B; ſo $mA = mB$.

178. *Axiom* 2. Theſe Magnitudes of which the ſame, or equal Magnitudes, are Equimultiples, are equal to one another.

179. *Axiom* 3. A *Multiple* of a greater Magnitude is greater than the ſame Multiple of a leſſer.

180. *Axiom* 4. That Magnitude whoſe Multiple is greater than the ſame Multiple of another, is greater than that other Magnitude.

181. *Axiom* 5. Magnitudes having the ſame Ratio to one and the ſame or equal Magnitudes, are equal to each other.

182. *Axiom* 6. Equal Magnitudes have to one and the ſame Magnitude the ſame Ratio: And the ſame unto equal Magnitudes has the ſame Ratio.

183. *Axiom* 7. Magnitudes to which one and the ſame Magnitude has the ſame Ratio, are equal.

184. *Axiom* 8. If two Magnitudes be referred to a third, that is the greateſt which has the greateſt Ratio.

185. *Axiom* 9. Ratios equal to one and the ſame Ratio, or to equal Ratios, are alſo equal to one another.

186. *Axiom* 10. Unequal Magnitudes, compared with one and the ſame or equal Magnitudes, cannot have equal Ratios.

187. THEOREM I. *Magnitudes have the ſame Ratio to one another which their Equimultiples have. Let* a *and* b *repreſent the two Magnitudes, and let* m *be the Number of Times each of theſe are contained in its Multiple; then,* ma, mb, *repreſent the Equimultiples, and what we affirm is, that* ma : mb :: a : b.*

For taking any Equimultiples of the firſt and third, *viz.* *cma*, *ca*, and any Equimultiples, *dmb*, *db*, of the ſecond and fourth; it is plain that if $cma = dmb$, as they are Equimultiples of *ca*, *db*, that
* 178. † 180. *ca** $= db$; and if $cma \Delta dmb$, then *ca* † $\Lambda\, db$; alſo if

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* It may be demonſtrated algebraically thus, The Quotient of *ma* by *mb* is $\frac{ma}{mb} = \frac{a}{b}$; and the Quotient of *a* divided by *b* is alſo $\frac{a}{b}$. ∵ $ma : mb :: a : b$. Q. E. D.

if cma Γ dmb, then ca Γ db. ∵ by Art. 156, as $ma : mb :: a : b$. Q. E. D.

188. THEOREM 2. *If four Quantities,* a, b, c, d, *are proportional,* viz. *as* a : b :: c : d; *then any Equimultiples of the two firſt will have the ſame Ratio as any Equimultiples of the two laſt,* viz. *as* ma : mb :: nc : nd.

For by Art. 187, As, $ma : mb :: a : b$,
And, as, $nc : nd :: c : d$.

But a is to b, in ſame Ratio as c to d by the Suppoſition, ∵ by Art. 185, as $ma : mb :: nc : nd$. Q. E. D.

189. THEOREM 3. *Triangles* ABC, ACD, *and Parallelograms* EC, CF, *of the ſame Altitude, are to one another as their Baſes,* BC, CD.* Pl. IV. F. 2.

Let BD be produced both Ways to the Points H, L, and let BG, GH, be each = BC; and DK, KL, each = CD; and let Lines be drawn as in the Figure. Then, becauſe the Δs AHG, AGB, ABC, have equal Baſes and the ſame Altitude, they are[a] = each other; ∵ whatever Multiple the Baſe HC is of the Baſe BC, the ſame Multiple is the Δ AHC of the Δ ABC. By the ſame Reaſon, whatever Multiple the Baſe LC is of the Baſe CD, the ſame Multiple

[a] 109.

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* The algebraic Demonſtration may ſtand thus. Let a = Baſe of the Parallelogram EC, b = Baſe of CF, and h = their Height; then ah = Area of EC, and bh = Area of CF. Now it is plain that as $ah : bh :: a : b$; for the Quotient of the firſt Term by the ſecond is $\frac{ah}{bh} = \frac{a}{b}$; and the Quotient of the third by the 4th is alſo $\frac{a}{b}$; ∵ it will be, as $ah : bh :: a : b$. Q. E. D. And as Δs are Halves of the Parallelograms, as the Parallelograms are as their Baſes, their Halves, *viz.* the Δs, muſt be in the ſame Ratio alſo. Q. E. D.

[e] 40. ∵ ∠BAD [e] = ∠E, and ∠DAC [f] = ∠ACE [a] =
[f] 72. ∠E; ∵ ∠BAD [g] = ∠DAC. Q. E. D.
[a] 59.
[g] 45.

193. THEOREM 6. *If four Lines, or Numbers, denoted by the Letters,* a, b, c, d, *are proportional;* viz. *as* a : b :: c : d; *the Rectangle contained by the Extremes is equal to the Rectangle under the Means,* viz. a × d = b × c. *And if the Rectangle contained by the Extremes be = the Rectangle contained by the Means, theſe four Quantities are proportional; that is, if* a × d = b × c, *then it will be, as* a : b :: c : d.*

Firſt. Since $a \times d$, and $b \times d$, are Equimultiples
[a] 187. of a and b, we have, as $a \times d : b \times d$ [a] :: $a : b$.
And ∵ ſince by the Suppoſition, $a : b :: c : d$,
[b] 185. we have $a \times d : b \times d$ [b] :: $c : d$. But $b \times c$, and
$b \times d$ are Equimultiples of c and d, ∵ $b \times c : b \times$
[c] 187. d [c] :: $c : d$; ∵ $a \times d : b \times d$ :: [d] $b \times c : b \times d$;
[d] 185. ∵ $a \times d$ [e] = $b \times c$. Q. E. D.
[e] 181.

Secondly. It is plain, that $a \times d$, $b \times d$, are E-
[f] 187. quimultiples of a and b; ∵ $a : b$ :: [f] $a \times d : b \times d$; and $b \times c$, $b \times d$, are Equimultiples of c, and d; ∵ $c : d :: b \times c : b \times d$; but $a \times d = b \times c$ by the Suppoſition, ∵ by writing $a \times d$, for its equal $b \times c$, we have, as $c : d :: a \times d : b \times d$; but it has been juſt ſhewn that $a : b :: a \times d : b \times d$, ∵ $c : d$ has the ſame Ratio as $a : b$, that is, as $a : b :: c\ d$. Q. E. D.

194.

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* This *Theorem* may be demonſtrated algebraically thus. Firſt,
[a] 156. $\frac{a}{b}$ [a] $= \frac{c}{d}$ ∵ multiplying by b we have $a = \frac{bc}{d}$, and this by d gives $da = bc$. Q. E. D. Secondly, If we divide each Side of the Equation $da = bc$ by d, we have $a = \frac{bc}{d}$, and this again by b gives $\frac{a}{b} = \frac{c}{d}$; ∵ by Art. 156, As $a : b :: c : d$. Q. E. D.

194. THEOREM 7. *Equiangular Triangles are ſimilar; that is, the Sides about the equal Angles are Proportionals.*

Pl. IV. F. 7.

Let D*a* = AB, D*b* = AC, then the ∠D being = ∠A by the Suppoſition, the △s D*ab*, ABC, are [a] equal in every Reſpect; ∵ ∠D*ab* = ∠ABC; but ∠ABC = ∠DEF by the Suppoſition, ∵ ∠D*ab*[b] = ∠DEF, ∵ *ba*[c] ∥ to EF; ∵ DE : D*a* :: [*] DF : D*b*. ∵ for D*a*, D*b*, writing their equals AB, AC, the Proportion becomes DE : AB :: DF : AC. Q. E. D.

[a] 58.
[b] 45.
[c] 40.
[*] 191.

And if F*d* be taken = CA, and F*c* = CB, it may be ſhewn in the ſame Manner, that FD : CA :: FE : CB. Q. E. D.

By joining theſe two Proportions in one, we have DE : AB :: DF : AC :: FE : CB. Q. E. D.

195. THEOREM 8. *If the outward Angle* CAE, *of a Triangle* ABC, *is biſected by a Line* AD, *cutting the Baſe* BC *produced in* D; *then the Segments* BD, DC, *will have the ſame Ratio as the other Sides of the Triangle*; viz. BD : DC :: BA : AC. *And if* BD : DC :: BA : AC, *the Line* AD *will biſect the outward Angle* CAE, viz. ∠CAD = ∠DAE.

Pl. IV. F. 8.

For let CB be ∥ to DA; then ∠CFA [a] = ∠DAE, and alſo ∠ACF [b] = ∠DAC; but the ∠DAE = ∠DAC by the Suppoſition, ∵ ∠CFA [c] = ∠ACF, ∵ AF [d] = AC. Now, as CF is ∥ to AD by the Suppoſition, it follows, BD : DC [e] :: BA : AF; but it has been juſt ſhewn that AF = AC, ∵ for AF writing its equal AC, the above Proportion becomes BD : DC :: BA : AC. Q. E. D.

[a] 40.
[b] 72.
[c] 45.
[d] 62.
[e] 191.

Secondly. By the Suppoſition, BD : DC :: BA : AC, and FC being ∥ to AD, BD : DC :: BA : AF, ∵ BA : AC [f] :: BA : AF; ∵ AC [g] = AF, and ∵ ∠AFC [h] = ∠ACF; alſo the ∠AFC [a] = DAE,

[f] 185.
[g] 181.
[h] 59.
[a] 40.

b 72. DAE, and ∠ACF [b] = ∠CAD, ∵ ∠DAE (=
c 45. ACF) [c] = CAD. Q. E. D.

Pl. IV. F. 9. 196. Theorem 9. *If two Triangles,* ABC, DEF, *have one Angle* A, *of one, equal to one Angle* D, *of the other, and the Sides containing the equal Angles proportional,* viz. AB : AC :: DE : DF, *the Triangles are equiangular and ſimilar.*

Let A*f* be = D*f*, and A*e* = DE; and let *fe* be
joined. Then the ∠A being = the ∠D by the
a 58. Suppoſition, the △s A*fe*, DFE, are [a] equal in
every Reſpect. By the Suppoſition, AB : AC ::
DE : DF; or, which is the ſame, by writing
equals, (*viz.* A*e* for DE, and A*f* for DF;) AB :
b 191. AC :: A*e* : A*f*. ∵ *fe* is [b] ∥ to BC, ∵ ∠A*fe* [c] =
c 40. ∠ACB, and ∠A*ef* = ∠ABC; ∵ the △s A*fe*,
ABC, are equiangular, and conſequently, as the △s
A*fe*, DFE, have been juſt ſhewn to be equal in
every Reſpect, ABC, DEF, are alſo equiangular;
d 194. and ∵ alſo [d] ſimilar. Q. E. D.

Pl. IV. F. 9. 197. Theorem 10. *If two Triangles,* ABC, DEF, *have their like Sides proportional, (thus,* AB : AC :: DE : DF; *and* AB : BC :: DE : FE.) *the Triangles are ſimilar to each other.*

Let A*f* = DF, and A*e* = DE, and let *fe* be
joined; then by the Suppoſition, AB : AC :: DE
: DF; and by writing equals, *viz.* A*e* for DE,
and A*f* for DF, the Analogy is, AB : AC :: A*e* :
a 191. A*f*; ∵ *fe* is [a] ∥ to BC. Again, by the Suppoſi-
tion, AB : BC :: DE : FE; and the △s ABC,
b 196. A*ef*. being [b] equiangular and ſimilar, AB : BC ::
c 185. A*e* : *fe*; ∵ by Equality of Ratios, DE : FE [c] ::
A*e* : *fe*; but A*e* = DE by the Conſtruction, ∵
writing DE for its equal A*e*, we have, DE : FE ::
DE : *fe*, conſequently FE = *fe*. Hence the three
Sides of the △s A*fe*, DFE, have been proved to be
reſpectively

reſpectively equal to each other, and $\because$ the $\triangle$s Afe, DFE, are [d] equal in all Reſpects; and conſequently, as the $\triangle$ Afe is ſimilar to the $\triangle$ABC, the $\triangle$DEF muſt be ſo too. Q. E. D. d 95.

198. THEOREM 11. *If four Quantities are proportional, they will alſo be Proportionals when taken inverſely*; viz. *if* a : b :: c : d, *then alſo*, b : a :: d : c.

For the Rectangle of the Extremes being * = the Rectangle of the Means, $a \times d = b \times c$; or, which is the ſame, $b \times c = a \times d$, $\because$ $b : a$ * $:: d : c$. Q. E. D. * 193.

199. THEOREM 12. *If four Quantities be proportional, they will alſo be proportional when taken alternately*; *that is*, *if* a : b :: c : d, *then* a : c :: b : d.

For the Rectangle of the Extremes being * = Rectangle of the Means, $a \times d = b \times c$, or, which is the ſame, $a \times d = c \times b$, $\because$ $a : c$ * $:: b : d$. Q. E. D. * 193.

200. THEOREM 13. *If four Lines, denoted by* a, b, c, d, *are Proportionals*, viz. a : b :: c : d; *they will alſo be compoundedly proportional*; *that is*, a + b : b :: c + d : d.†

Let AB $= a$, BC $= b$, AE $= c$, ED $= d$; then is AC $= a + b$, and AD $= c + d$. Becauſe by Pl. IV. Fig. 10.

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† Algebraically demonſtrated thus. For the Product of the Extremes, $\overline{a+b} \times d = ad + bd$, and the Product of the Means, $b \times \overline{c+d} = bc + bd$; but $a : b :: c : d$, by the Suppoſition; $\because$ $ad = bc$, conſequently $ad + bd = bc + bd$, or, which is the ſame, $\overline{a+b} \times d = b \times \overline{c+d}$, and $\because$ by Art. 193. $a + b : b :: c + d : d$. Q. E. D.

by the Suppoſition, $a : b :: c : d$, that is, AB : [a] 191. BC :: AE : ED ; BE, CD, are[a] ∥ to each other, and ∵ AC : BC :: AD : DE, that is, $a + b : b :: c + d : d$. Q. E. D.

201. THEOREM 14. *If four Lines, denoted by* a, b, c, d, *are Proportionals,* viz. $a : b :: c : d$, *they will alſo be Proportionals by Diviſion,* viz. $a - b : b :: c - d : d$.†

Pl. IV. Fig. 11. Let AC $= a$, CB $= b$, AD $= c$, DE $= d$; then, AB $= a - b$, AE $= c - d$; by the Suppoſition, $a : b :: c : d$, that is, AC : CB :: AD : [a] 191. DE, ∵ BD, DC, are[a] ∥ to each other, and ∵ AB : BC :: AE : ED; that is, $a - b : b :: c - d : d$. Q. E. D.

202. THEOREM 15. *If four Lines, denoted by* a, b, c, d, *are Proportionals,* viz. $a : b :: c : d$, *then they will alſo be Proportionals by Converſion, that is,* $a : a - b :: c : c - d$.*

Let AC $= a$, CB $= b$, &c. as in the laſt Theorem ; then BC, CD, being there ſhewn to be ∥ to each other, we have AC : AB[a] :: AD : AE, that is, $a : a - b :: c : c - d$. Q. E. D. [a] 191.

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† This may be demonſtrated analytically thus. The Product of the Extremes, $\overline{a-b} \times d = ad - bd$, and the Product of the Means $b \times \overline{c-d} = bc - bd$; but $ad = bc$ by Art. 193, conſequently $ad - bd = bc - bd$; or, which is the ſame, $\overline{a-b} \times d = b \times \overline{c-d}$; ∵ $a - b : b :: c - d : d$. Q. E. D.

* Algebraically thus. The Product of the Extremes $a \times \overline{c-d} = ac - ad$, and of the Means $\overline{a-b} \times c = ac - bc$; but it has been before ſhewn that $ad = bc$; conſequently $ac - ad = ac - bc$; and ∵ their Equals are equal to each other, that is, $a \times \overline{c-d} = \overline{a-b} \times c$, or Product of the Extremes = the Product of the Means, and ∵ theſe Quantities are proportional by Art. 193, as aſſerted in the Theorem. Q. E. D.

203. THEOREM 16. *If four Lines, denoted by* a, b, c, d, *are Proportionals,* viz. a : b :: c : d, *then it will alſo be,* a + b : a — b :: c + d : c — d. *This is uſually cited by the Word* mixtly. *

Let AE $= a$, EC and EG each $= b$, AD $= c$, DB and DF each $= d$, then AC $= a + b$, AG $= a - b$, AB $= c + d$, and AF $= c - d$. Since by the Suppoſition, $a : b :: c : d$, that is, AE : EC :: AD : DB; and AE : EG :: AD : DF; BC and GF are[a] || to ED, ∵ AC : AG :: AB : AF, that is, $a + b : a - b :: c + d : c - d$. Q. E. D. Pl. IV. Fig. 12. [a] 191.

204. SCHOLIUM. What has been, or may hereafter be demonſtrated, concerning Lines being Proportionals, muſt, it is manifeſt, hold good in any other Magnitudes; for Lines may be aſſumed, having the ſame Ratio to each other as the Magnitudes themſelves; and if the Ratios by which the Theorems are deduced are the ſame, the Concluſions muſt certainly be ſo likewiſe.

205. THEOREM 17. *If three Lines are Proportionals, the Rectangle under the Extremes is equal to the*

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* Analytically thus. The Product of the Extremes $\overline{a+b} \times \overline{c-d} = ac + bc - ad - bd$; and the Product of the Means $\overline{a-b} \times \overline{c+d} = ac - bc + ad - bd$. But it is known that $ad = bc$ by the Suppoſition and Art. 193; ∵ $+ bc - ad$ in one Equation, and $- bc + ad$ in the other, vaniſh, being equal to Nothing; whence the Equations become $\overline{a+b} \times \overline{c-d} = ac - bd$, and $\overline{a-b} \times \overline{c+d} = ac - bd$ alſo; conſequently $\overline{a+b} \times \overline{c-d} = \overline{a-b} \times \overline{c+d}$, that is, the Product of the Extremes = Product of the Means, and ∵ the Quantities proportional. Q. E. D.

the Square on the Mean; *that is, if* a : b :: b : c; *then* $a \times c = b^2$.

[a] 193. For, the Rectangle of the Extremes being [a] = the Rectangle of the Means, $a \times c = b \times b$; but the Rectangle $b \times b$ is a Square, $\because$ $a \times c = b^2$, Q. E. D.

Note. This follows ſo eaſily from Art. 193, that it might have been given as a Corollary thereto, but was not at that Time thought of.

206. THEOREM 18. *The Rectangles under proportional Lines are Proportionals. Thus, if* a : b :: c : d, *and* e : f :: g : h, *then we affirm that* $a \times e : b \times f :: c \times g : d \times h$.*

[a] 153. For, $a \times e$, and $b \times e$, being [a] Equimultiples of
[*] 187. a and b, we have $a \times e : b \times e$ * :: $a : b$; but $a : b :: c : d$ by the Suppoſition, $\because$ by Equality of Ra-
[†] 185. tios, $a \times e : b \times e$ † :: $c : d$. Again, $c \times g$, $d \times g$, are Equimultiples of c, d, $\because$ $c \times g : d \times g :: c : d$; $\because$ by Equality of Ratios, $a \times e : b \times e :: c \times g : d \times g$. Alſo $b \times e$, $b \times f$, are Equimultiples of e, f, $\because$ $b \times e : b \times f :: e : f$; but $e : f :: g : h$, by the Suppoſition, $\because$ by Equality of Ratios, $b \times e : b \times f :: g : h$.

Once again; $d \times g$, $d \times h$, are Equimultiples of g and h, $\because$ $d \times g : d \times h :: g : h$. $\because$ $b \times e : b \times f :: d \times g : d \times h$, by Equality of Ratios. By Alternation the Analogy (above found) $a \times e : b \times e ::$

$c \times g$

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* This may be thus demonſtrated by *Algebra*, *viz.* By the Suppoſition, as $a : b :: c : d$; and $e : f :: g : h$; hence the Product of the Extremes being * = the Product of the Means, we
[*] 193. have $ad = bc$, and $eh = fg$; $\because$ by multiplying each Side of the firſt Equation by the correſpondent Term of the other Equation, we get $aedh = bfcg$, (becauſe equal Things multiplied by equal Things muſt produce equal Products,) or, which is the ſame, $ae \times dh = bf \times cg$; $\because$ (by Art. 193.) as $ae : bf :: cg : dh$, Q. E. D.

$c \times g : d \times g$, becomes $a \times e : c \times g :: b \times e : d \times g$, and the Analogy juſt found, taken alternately, is $b \times e : d \times g :: b \times f : d \times h$; ∵ by Equality of Ratios, $a \times e : c \times g :: b \times f : d \times h$. Q. E. D.

207. THEOREM 19. *The Squares upon four proportional Lines are proportional*; viz. *if* $a : b :: c : d$; *then we affirm that* $a^2 : b^2 :: c^2 : d^2$.

For, if $a : b :: c : d$, and $a : b :: c : d$, then by the laſt Theorem, $a \times a : b \times b :: c \times c : d \times d$. Q. E. D.

208. THEOREM 20. *If four Squares are Proportionals, their Sides will be ſo too*; *that is, if* $a^2 : b^2 :: c^2 : d^2$, *then we affirm that* $a : b :: c : d$.

For let $a : b :: c : e$, then *per* laſt Theorem, $a^2 :$
$b^2 :: c^2 : e^2$, ∵ by Equality[*] of Ratios, $c^2 : d^2 :: c^2 :$ * 185.
e^2, ∴ d^2[*] $= e^2$. ∵ $d = e$, ∵ by writing d for its * 183.
Equal e, we have $a : b :: c : d$. Q. E. D.

††† We may perhaps treat more fully of the Doctrine of Proportionals in a future *Eſſay*, in a new and algebraic Manner; our Intention at this Time being only to give ſo much as is in ſome Degree neceſſary for the preſent Occaſion: In doing which we have been obliged to deviate greatly from *Euclid's* Method; and indeed, after all that has or can be done, to reſtore the genuine Demonſtrations of *Euclid*, that Method will be found generally too obſcure for Learners at their firſt ſetting out. And we hope thoſe who are prejudiced to Antiquity will excuſe us in ſaying, that, as Mathematics are now improved, we cannot ſee any Neceſſity of adhering to *Euclid's* Demonſtrations of the Doctrine of Proportionals. And to ſpeak plainly, we think many of the Propoſitions are in a Manner uſeleſs to an Algebraiſt, as many Theorems which are with Difficulty

demonſtrated by them, may be eaſily inveſtigated by the analytic Art.

Notwithſtanding what Truth has obliged us to remark, we may venture to affirm, that we hold *Euclid* in the higheſt Veneration, and, if we are not miſtaken, more juſtly than thoſe who are ſo much bigotted to him, as to condemn any Author, who ſhall dare to deviate from him in the leaſt Article. The Cenſures of ſuch will therefore give us no Concern.

209. THEOREM 21. *If in a right-angled Triangle*
Pl. IV. ABC, *right-angled at* A, *a Perpendicular* AD *be let*
F. 13. *fall from the right Angle, to the oppoſite Side* BC; *the Triangles on each Side of it ſhall be ſimilar to the whole Triangle, and to one another.*

For the △ADC being right-angled at D, the ∠
[a] 83. CAD is[a] = a Rt. ∠ — ∠C, and the △ABC being Rt. ∠d at A, the ∠B = a Rt. ∠ — ∠C; ∵ ∠
[b] 45. CAD[b] = ∠B; and ∠C being common to both △s, and ∠s ADC, BAC, Rt. ∠s, the △s ABC,
[c] 194. ADC, are equiangular, and ∵[c] ſimilar. In the ſame Manner it may be ſhewn, that the △s BAD, ABC, are ſimilar; and the △s ADC, ABD, being each ſimilar to the △ABC, they muſt conſequently be ſimilar to each other. Q. E. D.

210. COROLLARY 1. *Hence theſe Analogies.*

1. AB : BC :: AD : AC.
2. BD : DA :: DA : DC. *
3. BC : AB :: A : BD.
4. BC : AC :: AC : DC.

211. COROLLARY 2. *If from any Point* A, *in the Circumference of a Circle, the Perpendicular* AD *is let*

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* This is all we mean when we ſay AD is a geometric mean Proportional between BD and DC.

let fall on the Diameter BC, *and the Chords* AB, AC, *drawn from the Point* A *to the Extremities of the Diameter* BC; *as the Angle in a Semicircle is a* * *right* ∠, * 148. *it plainly follows, that the Rectangle of the two Chords is equal to the Rectangle under the whole Diameter and that Perpendicular; alſo that the Rectangle under the two Segments of the Baſe is equal to the Square of the Perpendicular; and laſtly, that the Square of either Chord is equal to the Rectangle under the whole Diameter and adjacent Segment.**

For in four proportional Quantities, the Rectangle of the Extremes being [a] = the Rectangle of the Means, [a] 193. from the Analogies in Art. 210 we have,

$$\left.\begin{array}{l} 1.\ AB \times AC = BC \times AD. \\ 2.\ BD \times DC = DA^2. \\ 3.\ BC \times BD = AB^2. \\ 4.\ BC \times DE = AC^2. \end{array}\right\} \text{Q. E. D.}$$

212. THEOREM 22. *If two Lines cut one another within a Circle, (as in* Pl. IV. Fig. 14.) *the Rectangle of the Segments of one of them is equal to the Rectangle of the Segments of the other,* viz. AE × EC = BE × ED. *And if two Lines cutting a Circle, being produced, interſect each other without the Circle, (as in* Pl. IV. Fig. 15.) *then alſo will* AE × EC = BE × ED.

For let BC, AD, be joined; then the ∠s C, and D, ſtanding on the ſame Arc AB are [a] equal to [a] 146. each

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* From hence may be deduced a very eaſy Demonſtration of the celebrated Propoſition generally cited by 47 *Euclid's 1ſt Book*, *viz.* our 115th Article. For, by this, $BD \times BC = AB^2$, and $DC \times BC = AC^2$; ∵ $BD \times BC + DC \times BC$ (by the 46 Art.) $= AB^2 + AC^2$; or, which is the ſame, $\overline{BD + DC} \times BC = AB^2 + AC^2$; that is, ($BC \times BC$, or) $BC^2 = AB^2 + AC^2$. Q. E. D.

[b] 69. each other; and the ∠BEC [b] = ∠AED, the ∆s
BEC, AED, have two ∠s of one reſpectively =
[c] 80. two ∠s of the other, and ∵ the third ∠A [c] = the
third ∠B; conſequently, the ∆s are equiangular,
[d] 194. and [d] ∵ ſimilar. Hence, AE : ED :: BE : EC; ∵
[e] 193. AE × EC [e] = BE × ED.

Pl. IV. 213. THEOREM 23. *If from any Point* C, *with-*
Fig. 16. *out a Circle, two Lines* CA, CB, *are drawn in ſuch Manner, that* CB *touches the Circle, and* CA *cuts it; then will* CA × CD = BC^2.

[a] 151. For let BD, BA be joined; then the ∠DBC [a] =
A, and the ∠C common to both ∆s, ∵ the ∆s DBC, CBA, have two ∠s in one = two ∠s in the other, ∵ the remaining ∠BDC in one = the remaining ∠CBA in the other, ∵ the ∆s are equi-
[b] 194. angular, and conſequently [b] ſimilar, ∵ AC : CB ::
[c] 193. CB : CD; ∴ AC × CD [c] = CB × CB. Q. E. D.

Pl. IV. 214. THEOREM 24. *In equal Circles, Angles,*
F. 17. *whether at the Centers or Circumferences, have the ſame Ratio with the Circumferences on which they ſtand. Thus, let* G *and* H *be the Centers of the two Circles; then we affirm, that, as the Circumference* BC : *the Circumference* EF, :: *the* ∠ BGC : ∠ EHF *and* :: ∠ BAC : ∠ EDF.

For if we take any Number of Circumferences, CK, KL, each = BC; and any Number whatever FM, MN, each = EF; and join GK, GL, HM, HN; then, becauſe the Circumferences BC, CK, KL, are all equal, the ∠s BGC, CGK, KGL, will be = each other, (for 'tis manifeſt they would exactly cover each other;) ∵ whatever Multiple the Circumference BL is of the Circumference BC, the ſame Multiple is the ∠BGL of the ∠BGC. And by the ſame Method of Reaſoning the Circumference EN is the ſame Multiple of the Circumference EF, as the ∠EHN is of the ∠EHF. And if the Circumference BL be = the Circumference EN, the ∠ BGL will be = ∠EHN, (as they would exactly cover each other;) and if the Circumference BL be Γ EN, the ∠BGL will be Γ ∠EHN, and if BL be Λ EN, the ∠ BGL Λ ∠ EHN. Here, then, are four Magnitudes, *viz.* the two Circumferences BC, EF, and the two ∠s BGC, EHF; and of the Circumference BC, and of the ∠BGC, have been taken any Multiples whatever, *viz.* the Circumference BL, and ∠BGL; and of the Circumference EF, and ∠EHF, any Equimultiples whatever, *viz.* the Circumference EN, and ∠EHN; and it has been proved, that if the Circumference BL be either Γ, =, or Λ the Circumference EN, the ∠BGL will be reſpectively Γ, =, or Λ the ∠ EHN; ∵ the Circumference BC : Circumference EF* :: ∠BGC : ∠EHF. Q. E. D. * 156.

As to the ∠s BAC, EDF, at the Circumferences, they are the [b] Halves of the ∠s at the Centers BGC, EHF; ∵ as the Wholes have been ſhewn above to be in Proportion to the Circumferences BC, EF, their Halves muſt be ſo too. Q. E. D. [b] 145.

215. Corollary 1. *The* ∠BGC *in the Center* : 4 Rt. ∠s :: *the Circumference or Arc* BC *on which it ſtands* : *the whole Circumference of the Circle.*

216. Corollary 2. *Hence the Circumferences* DE, BC, *of unequal Circles, which ſubtend equal Angles at the Centers, as* ∠*s* DAE, BAC, *are ſimilar*; viz. *as Arc* DE : *the whole Circumference of the Circle* DEF :: BC : *the Circumference of the Circle* BCG. *And if they are ſimilar, they will ſubtend equal Angles.* Pl. IV. F. 18.

For 1ſt. ∠DAE : 4 Rt. ∠s [a] :: DE : the Circumference of the whole Circle DEF; and (∠BAC, or, which is the ſame,) ∠ DAE : 4 Rt. ∠s :: Arch BC : the Circumference of the whole Circle [a] 215.

BCG. ∵ by Equality of Ratios, Arc DE : Circumference of the Circle DEF :: Arc BC : Circumference of the Circle BCG. Q. E. D.

Secondly. As ∠ DAE : 4 Rt. ∠s[a] :: the Arc DE : the Circumference DEF; and ∠ BAC : 4 Rt. ∠s :: Arc BC : the Circumference of Circle BCG; but, by the Suppoſition, the Arc DE : the whole Circumference of the Circle DEF :: BC : the whole Circumference of the Circle BCG; ∵ by Equality of Ratios, ∠DAE : 4 Rt. ∠s :: ∠BAC : 4 Rt. ∠s; ∵ ∠DAE[b] = ∠BAC. Q. E. D.

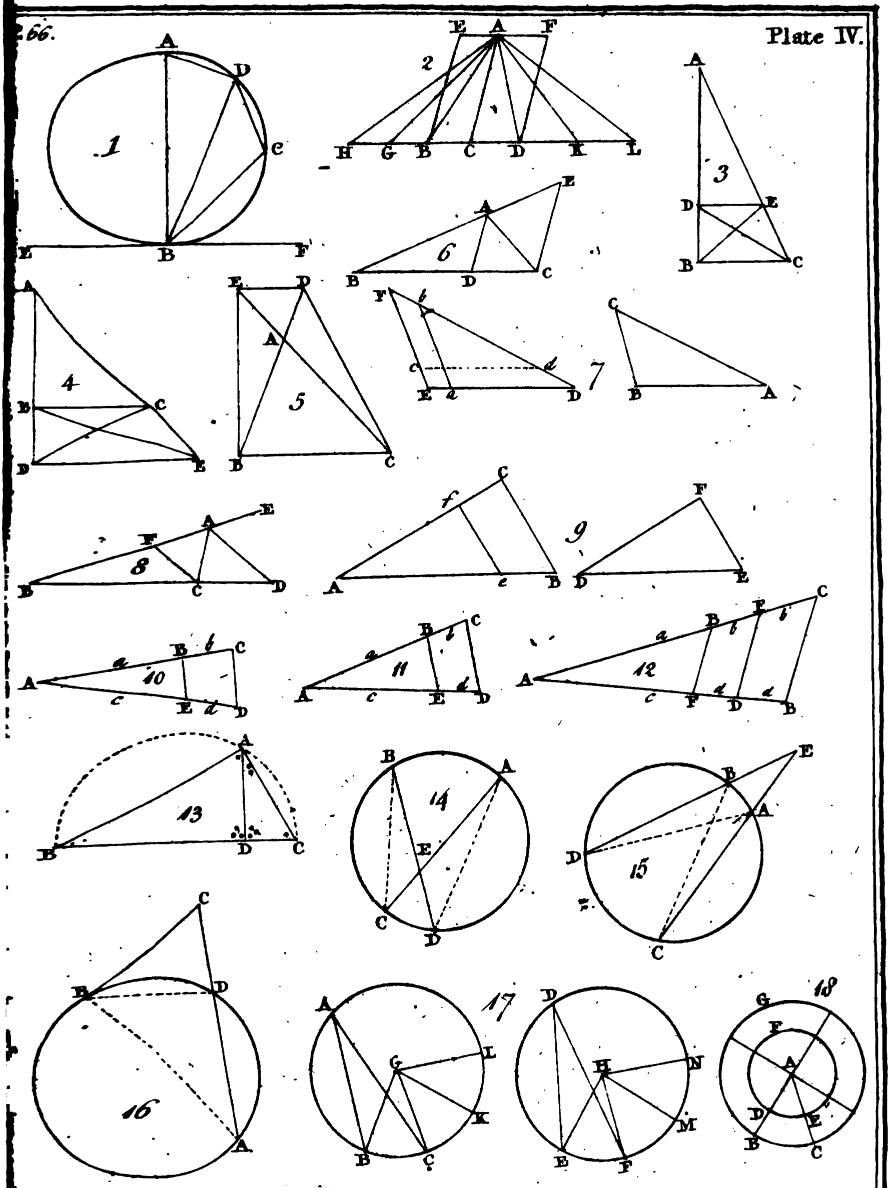
[a] 215.

[b] 181.

217. *Scholium.* Mathematicians, for the more eaſy comparing the Proportion which Angles bear to each other, ſuppoſe the Circumference or Periphery of every Circle to be divided into 360 equal Parts, which they call Degrees, (ſo that 180 Degrees are a Semicircle, or 2 Rt. ∠s, and 90 Degrees a Quadrant, or Rt ∠;) each Degree is ſuppoſed to be divided into 60 equal Parts, called Minutes, and each Minute into 60 equal Parts, called Seconds, &c. —— Hence it is plain, that in unequal Circles, as well as in equal, equal Angles are ſubtended by an equal Number of Degrees. For, as a Degree is $\frac{1}{360}$ Part of the Circumference of any Circle, the Length of a Degree will increaſe, or decreaſe, in the ſame Proportion as the Circumferences of the Circles; and ſo the Arch, repreſenting any Number of Degrees in one Circle, will be to the whole Circumference, as the Arch repreſenting the ſame Number of Degrees in any other Circle, to the whole Circumference of that other Circle, and ∵ will, by the laſt Corollary, ſubtend equal ∠s. Hence appears the Reaſon of the Method of denoting an ∠ by ſaying it contains ſo many Degrees.

BOOK

Plate IV.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BOOK V.

Of PROBLEMS.

218. IT will now be convenient for the Learner to procure a Caſe of Pocket Inſtruments, for his more eaſy performing the following Problems.

The Caſe we would recommend ought at leaſt to contain 2 Pairs of Compaſſes, a Drawing-Pen and Point, a Pencil and Point, Parallel-Ruler, Ivory Plotting-Scale, Ivory Sector, and Braſs Protractor. To theſe may alſo be added, a Pricking or Dotting-Wheel-Point, a Protracting-Pin, a Pair of Bows, Tracing-Point, Feeder, &c.

219. *Compaſſes* are ſo well known to every one, that it would be loſt Labour to deſcribe them; however, it may be of Uſe to ſome Perſons to hint, that one of the Compaſſes hath one Leg to take out, that in its Stead may be occaſionally uſed, the Drawing-Pen-Point, a Pencil-Point, and Dotting-Wheel-Point. ——— In chooſing Compaſſes take Notice, that in opening and ſhutting, the Legs move not by Starts, but that the Motion in the Joint be * eaſy and regular; and that the Points are of an equal Length, &c. For leſſening the irregular Friction, (if any,) the Head of the Compaſſes may be warmed, and a little || yellow Wax melted and poured between the Joints.

The

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* And for this End let Part of the Joint be Steel, for Braſs and Steel will wear more equable than either ſingly. Alſo let the Pin or Axle of the Joint be a Steel Screw, by Means of which you can make your Compaſſes to move in the Joint ſtiffer or eaſier at Pleaſure.

|| Or, if the Wax is too ſtiff of itſelf, a little Wax and Oil mixed together.

The Solder for cementing the Steel Points to the Braſs, is generally made with twice as much Silver as Copper; which muſt be melted in a Crucible, and, when cold, hammered to about the Thickneſs of a Card. Solder is alſo frequently made by melting 3 Parts of Copper with one of Zink.

In Order to make the Solder run the better between the Parts to be ſoldered, Borax, bruiſed very fine, is laid on the Parts to be cemented, and melted together with the Solder.

To one Point of the ſmaller Compaſſes is ſometimes fixed in the Shank, a Spring, which, by Help of a Screw, when you have opened the Compaſs nearly to the required Diſtance, moves it very regularly and gradually, without any ſtarting to the Point intended, for the more accurately taking off any required Diſtance. Thus, holding the Compaſſes in the left Hand, (with the Screw turned towards the right,) if with the other you turn the Screw backward, the Spring Point will approach nearer to the other Point; but if you turn the Screw forward, the Spring Point will be farther diſtant from the other: However, after the Learner has been ſome Time uſed to a Pair of Compaſſes, he will not ſtand in much Need of this Contrivance.

The leſſer Compaſſes are uſed in transferring the Meaſures of Diſtances from one Place to another, or in deſcribing Circles, or Parts of Circles. ——— The larger Compaſſes are chiefly intended for deſcribing Circles, or Parts of Circles, either obſcurely by the Steel Point, in Ink by the Drawing-Pen-Point, in Lead by the Pencil Point, or dotted by the Dotting-Wheel-Point; for either of theſe, as before obſerved, may be fixed in the Shank of the Compaſſes. The *Dotting-Wheel-Point* is not abſolutely neceſſary, for dotted Lines are better made by the Drawing Pen or Point, (though ſomething ſlower,) for it is but ſeldom the Dotting-Wheel-Point is ſo well made as to perform accurately.

As almoſt every Learner, at his firſt Beginning, handles a Pair of Compaſſes very aukwardly, it may be proper to caution him by all means to endeavour to make Uſe of them with the right Hand only, never uſing the left, unleſs in a Caſe of Neceſſity, which after ſome Practice he will ſeldom have Occaſion for.——In uſing the Compaſſes be always careful to keep them nearly upright.

The *Bow* is a ſmall Kind of Compaſſes, one of whoſe Legs is as common, the other is a Drawing Pen Point: Its Uſe is to deſcribe Arcs or Circles of ſhort Radii, which could not be ſo readily and accurately done by the large Compaſſes and Drawing-Pen-Point.

The *Drawing Pen Point* and *Pencil Point* (or Crayon Point, as it is frequently called,) in the beſt Caſes have generally a Socket fitted to them, that they may fill up but one of the Holes or Partitions of the Caſe. Theſe Points, and alſo the Dotting-Wheel-Point, have a Joint in them juſt under that Part which locks into the Shank of the Compaſſes, that the Part below the Joint may ſtand perpendicular, or nearly ſo, to the Paper, or Plane, on which the Arc is to be deſcribed; otherwiſe it will not be ſmooth.

Of the larger Compaſſes, thoſe are eſteemed the beſt, whoſe moveable Points are locked in by a Spring and Catch fixed in the Shank; for if this Spring be well effected, the Point is thereby kept firm and ſteady; the contrary of which frequently happens when the Point is kept in by a Screw in the Shank.

If, inſtead of the larger Compaſſes being made with ſhifting Points, there were two Pairs put into the Caſe, to one of which the Ink Point was fixed, and to the other the Pencil Point, this, as Mr. *Robertſon* juſtly * obſerves, would ſave the Trouble of changing the Points in the Compaſs at every Time they were uſed, and the additional Expence, or Bulk of the Caſe, would be very little.

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* In his Treatiſe of Math. Inſtruments, &c.

The *Black-lead Pencil* is of great Uſe to deſcribe the firſt Sketch of a drawing before it is marked with Ink; becauſe any falſe or ſuperfluous Lines may be rubbed out with a Piece of ſtale Bread.

Black-Lead is produced in many Countries; but the beſt yet diſcovered is found in the North of *England*; it is dug out of the Ground in Lumps, and ſawed into Scantlings proper for Uſe. If it will not bear paring to a Point with a ſharp Penknife, it is not fit for drawing on Paper. There are three Sorts of Black-lead; the ſofteſt is proper for taking rough Sketches, the middling for drawing of Landſcapes and Ornaments, and the hard for drawing mathematical Figures.

The *Feeder* is a thin flat Piece of Braſs, or Silver, commonly fixed to a Cap that is put on the Pencil: Its Uſe is either to put Ink between the Blades of the Drawing Pen, or to paſs it between the Points when the Ink by drying does not flow freely.

The *Drawing-pen* is uſed only for drawing right Lines: It conſiſts of two Blades with Steel Points, fixed to a Handle; the Blades being a little bent, by Means of a Screw, cauſe the Steel Points to approach as near together as you pleaſe, ſo that a Line may be drawn ſtronger or finer at Pleaſure. When you uſe this Inſtrument put the Ink between the Blades, either with a common Pen or the Feeder, and by moving the Screw bring the Points to a proper Diſtance; then hold the Pen a little inclined, but take Care that both Points touch the Paper, by which Means you may draw a neat Line equally broad in all Parts; which is not eaſily done by a common Pen. ——— Before you put the Pen into the Caſe after uſing, it will be neceſſary to clean the Points from the Ink, otherwiſe they will be inclined to ruſt, &c. If one of the Blades is made to move on a Joint, as the *Bow* generally is, they would be more eaſily cleaned.

The *Protracting-pin* is a ſmall Piece of ſharp Steel much reſembling the Point of a Needle, fixed into the lower End of the Top Part of the Handle of the *Drawing-pin*; it ſcrews for Preſervation into the lower Part of the Handle: Its Uſe is to ſet off Angles from the Protractor, (as will be ſhewn hereafter,) and to mark off the Interſection of Lines, that the Places of croſſing may be the better ſeen.

220. We think it needleſs to deſcribe the *Tracing Point* in this Place, as it will come in more properly in another; and the Conſtruction of the other Inſtruments will be ſhewn, as they occaſionally become neceſſary or uſeful. We ſhall therefore, in this Place, only obſerve, that as a Ruler muſt be very ſtrait to draw right Lines with, it may be examined by drawing a right Line with it, and then turning the Ruler about End for End, apply the ſame Side to the Line, and, if the Ruler exactly coincides with the Line, the Ruler is ſtrait, but if not, you will ſee where the Fault is. A good Eye will alſo diſcover whether a Ruler is ſtrait by only looking over the Edge of it.

221. PROBLEM I. *To draw a Line from a given Point* C, *equal to a given Line* AB. P.V.F.1.

Take a Pair of Compaſſes, and placing one Point in A, open the other Point to B; then will the Diſtance between the Points of the Compaſſes Legs be = the given Line AB; ∵ ſetting one Leg in the Point C, with the other make another Point D, and by placing the Edge of a Ruler to touch the Points C, D, draw the right Line CD, and it is manifeſt the Thing is done; *viz.* CD = AB.

222. PROBLEM 2. *From a greater Line,* AB, *to cut off a Part equal to a leſſer Line* C. Fig. 2.

This may be eaſily done by opening the Compaſſes equal to the Length of the Line C, (as ſhewn in the 1ſt Problem,) and laying that Diſtance from A to D on the Line AB, ſo is AD = the Line C. Which was to be done.

Fig. 3. 223. PROBLEM 3. *About a given Point,* A, *to deſcribe a Circle whoſe Radius ſhall be equal to a given Line* B.

Open the Compaſſes = the Line B; then reſting the Point of one Leg in A, carry the other about from where it falls, as C, till it arrives again at that Point, and it is manifeſt the required Circle will be deſcribed by the moving Point.

N. B. In deſcribing of Circles be careful not to preſs the Paper, either at the Center or under the moveable Point, with more than juſt the Weight of the Compaſſes, to avoid making great Holes or Blots, which frequently happen to young Perſons at their firſt beginning to practice.

Fig. 4. 224. PROBLEM 4. *To make a Triangle whoſe Sides ſhall be reſpectively equal to three given Lines.*

Let the three given Lines be A, B, C; take DE [a] = C, then about D, as a Center, with a Radius = B; deſcribe Part of [b] a Circle, *viz.* Arc *ab*, and take DE = C, and about E, with a Radius = A, deſcribe Part of the Circumference of a Circle, *viz.* the Arc *dc*; then from the Point of their Interſection draw the Lines FD, FE, and the △DFE will be that required; for by the Conſtruction, DE = C, DF = B, EF = A.

[a] 221.
[b] 223.

Note. A + B muſt be ⊏ C, otherwiſe the Arcs will not interſect, and ſo the Problem would be impoſſible.

225.

225. Problem 5. *To biſect a given Angle* BAC, *that is, to divide it into two equal Angles.* Pl. V. F. 5.

Construction. With any Radius AD, ſweep the Arc DE; then with any Radius Γ ½ the Diſtance of the two Points D, E, about the Points D and E, as Centers, ſweep the Arcs *cd*, *ab*, and from the Point of Interſection F, draw the Line AF, and it is done; *viz.* the ∠FAC = ∠FAB.

Demonstration. AD being = AE, and DF = EF, by the Conſtruction, and AF common to both △s DAF, EAF, the △s are themſelves [a] equal in all Reſpects; ∵ ∠DAF = ∠EAF. Q. E. D. [a] 95.

226. Problem 6. *To biſect a given finite Line* AB, *that is, to divide it into two equal Parts.* Pl. V. F. 6.

Construction. About the Points A and B, as Centers, with any Radius Γ ½ AB, deſcribe Parts of Circles, *viz.* the Arcs *ab*, *cd*, and join the Points of Interſection E, F; and the Line EF will cut the Line AB in G, the Point which was required.

Demonstration. For by the Conſtruction, AE = AF = BE = BF, and EF common; ∵ the △s EAF, EBF, have 3 Sides of one reſpectively = 3 Sides of the other, and ∵ the △s are themſelves [a] equal in all Reſpects, and ∵ ∠AEG = ∠ BEG. Hence the △s AEG, BEG, have two Sides AE, EG, of one = two Sides BE, EG, of the other, and the contained ∠ AEG of one = the contained ∠BEG of the other, ∵ the Baſe AG [b] = the Baſe BG. Q. E. D. [a] 95. [b] 58.

227. Corollary. *This Problem ſerves alſo to raiſe a Perpendicular on the Middle of a Line.*

For we have juſt ſhewn that the Point G is in the Middle of the Line, *viz.* AG = BG; alſo AE = BE, and EG common; ∵ the △s AEG, EBG,
a 95. are [a] equal in all Reſpects, and ∵ ∠AGE = ∠
c 14. BGE; ∵ EG [c] ⊥ to AB. Q. E. D.

Pl. V. F. 7. 228. PROBLEM 7. *To draw a Perpendicular to a given Line* AB, *from any Point* C *in that Line.*

CONSTRUCTION. From the Point C ſet off any Diſtance towards A, as CD; and from the ſame Point the ſame Diſtance CE towards B. Then with any Extent of the Compaſſes r CD, putting one Leg in D, with the other ſweep the Arc *ab*, and with the ſame Extent of the Compaſſes, placing one Leg in E, with the other ſweep the Arc *cd*; then from the Point of Interſection F, draw the Line FC, and it is the ⊥ which was required.

DEMONSTRATION. CD being = CE, and DF = EF, by the Conſtruction, and FC common, the △s DCF, ECF, have 3 Sides of one reſpectively = the three Sides of the other; and ∵ the △s themſelves are equal in all Reſpects, and conſequently the ∠DCF = ∠ECF. Q. E. D.

Pl. V. F. 8. 229. PROBLEM 8. *To draw a Perpendicular to a given Line* AB, *of an unlimited Length, from a given Point* C, *without it. Or, in other Words, to let fall a Perpendicular from a Point above to a Line beneath.*

CONSTRUCTION. With any Radius of a ſufficient Length to interſect the Line AB, with one Leg of the Compaſſes in C, with the other ſweep the Arc DE; then with any Extent of the Compaſſes r $\frac{1}{2}$ DE, one Leg reſting in D, with the other ſweep the Arc *ab*; in like Manner, putting one Leg in E, with the other ſweep the Arc *cd*, then from the Point of Interſection, F, draw a Line to C, and it will be the $\perp$. Q. E. I.

DEMONSTRATION. For DC being $=$ CE, and DF $=$ EF, by the Conſtruction, and CF common, the $\triangle$s DCF, ECF, have 3 Sides of one reſpectively
$=$ 3 Sides of the other, and $\because$ they are [a] $=$ each [a] 95.
other in all Reſpects: Conſequently the $\angle$ DCG $=$ $\angle$ ECG. Now DC being $=$ EC, GC common, and the contained $\angle$DCG $=$ the contained $\angle$ECG, as hath been juſt ſhewn, the $\triangle$s DGC, EGC,
are [b] $\doteq$ each other in all Reſpects; and conſequently [b] 58.
the $\angle$DGC $=$ $\angle$EGC. Q. E. D.

Otherwiſe thus.

CONSTRUCTION. Join A, C, and [a] biſect the Line [a] 226.
AC in E; then with the Radius EA, or EC, ſweep Pl. V.
the Semicircle CDA, and where it interſects the Line Fig. 9.
AB in D, draw the Line CD, and it will be that required.

DEMONSTRATION. For the $\angle$ ADC in a Semicircle is a [*] Rt. $\angle$. Q. E. D. [*] 148.

230. PROBLEM 9. *At a given Point* A, *in a given Line* AB, *to make an Angle equal to a given Angle* EDF. Pl. V. Fig. 10.

CONSTRUCTION. About the Points A and D, with any Radius, deſcribe the Arcs *ab*, *cd*, then take with the Compaſſes the Diſtance between the two Points, *d*, *c*, which ſet off from *b* to *a*, on the Arc *ab*, and thro' the Point *a* draw the Line AC; then is the $\angle$BAC $\doteq$ $\angle$EDF.

DEMONSTRATION. For let the Points *d*, *c*, and *b*, *a*, be joined; then by the Conſtruction the 3 Sides of the $\triangle$ *b*A*a* are reſpectively $=$ the 3 Sides of the $\triangle$

*d*D*c*, and ∵ the △s themſelves are equal in all Reſpects; conſequently the ∠A = ∠D. Q. E. D.

231. Problem 10. *To conſtruct and ſhew the Uſe of a Protractor, that is, a Semicircle divided into Degrees, for ſetting off or meaſuring Angles.*

Pl. V. Fig. 11.

Construction. Let ADB be a ſemicircular Piece of Braſs. Now to divide the Circumference ADB into 180 equal Parts, called Degrees, we proceed thus. On the Center C ⊥ AB draw [a] CD, ſo will the ∠ ACD or BCD be a Rt. ∠, or the Arc AD = 90 d. = BD. Then take the Radius AC in the Compaſſes, and ſet it from A to E, and from B to F, then is AE, EF, FB each = 60 Degrees; and ſetting the ſame Diſtance from D to G, and from D to H, the Arcs DG, DH, are each = 60 d. and ſo AG, GE, ED, DF, FH, HB, each = 30 Degrees. Hence, by dividing each of theſe Arcs into 3 equal Parts, (by a Trial or two, or by Art. 241.) the Circumference will be divided into every 10 Degrees; which by taking the Halves will be divided into every 5 Degrees, and by a Trial or two the 5th Part of theſe laſt Diviſions may be taken, and ſo the Semicircle divided into every ſingle Degree, as repreſented in the Figure.

[a] 228.

Demonstration. The only Thing requiring a Demonſtration in this Conſtruction is, that the Radius is = the Chord of 60 Degrees; or, that the Compaſſes being opened = the Radius, one Leg reſting in the Point A, the other will fall on the Circumference in E, the Point of 60 Degrees. For ſuppoſe A, E, joined, then by the Conſtruction, the △ACE would be equilateral, and ∵ [a] equiangular; ∴ the ∠ ACE = [b] $\frac{1}{3}$ of 2 Rt. ∠s, that is, $\frac{1}{3}$ of 180 Degrees = 60 Degrees. Q. E. D.

[a] 60.
[b] 85.

232. *The 1ſt Uſe of the* PROTRACTOR *is, to make an Angle equal to any Number of Degrees.*

For Example, at the Point C, on the Line IC, to make an ∠ = 60 Degrees. Lay the Diameter AB on the Line IC, ſo that the Center of the Protractor may be at the angular Point C; then againſt 60 Degrees, (*viz.* the Point E,) make a * Dot, and, taking off the Protractor, draw the Line CK, to paſs thro' the Point E, and the ∠ KCI will be that required.

233. *Secondly. To meaſure an Angle.*

This is only the Reverſe of the laſt Article, and is done by laying the Protractor as there directed, and it will be ſeen by Inſpection what Number of Degrees is cut by the Line KC. —— All this is very plain.

234. PROBLEM II. *To conſtruct and ſhew the Uſe of a Line of Chords.* Pl. V. Fig. 12.

Definition. A Line of Chords is only the Chords taken from a Quarter of a Circle and ſet off on a Rt. Line.

Hence this CONSTRUCTION. Set one Point of the Compaſſes in the Point A, Fig. 11, and open the other Leg to any Number of Degrees, for Example Sake, to G, 30 Degrees; then that Diſtance laid from *a* towards *c*, on the Line *ac*, Fig. 12. will give the Point *b* for 30 Degrees. After the ſame

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* For this there is in the Drawing Pen a Point like a Pin, called a *Protracting Pin.* Another Kind of Protractor is made by transferring the Degrees from the Arch to the Sides of a Scale, or Ruler, as is intended to be fully deſcribed in another *Eſſay.*

Manner the other Degrees as far as 90 are laid on the Line *ac*, which then is called a Line of Chords. This is too manifeſt to need a Demonſtration.

235. *The Uſe of a Line of Chords* is very eaſy. For 1ſt, *To make an Angle equal to any Number of Degrees*, the Chord of 60 Degrees being = the Radius, as demonſtrated in the laſt Problem, with that Extent, (commonly called a Sweep of 60) deſcribe an Arc of a Circle of a ſufficient Length; then take off from the Scale (Fig. 12.) the Chord of the Number of the Degrees the ∠ is to contain, and ſet off that Diſtance on the Arc; then draw a Line from the angular Point, as in Art. 232, &c. *To meaſure an Angle being only the Reverſe of this*, we ſhall omit particular Directions as unneceſſary.

Note. If the ∠ is more than 90 Degrees, firſt ſet off the Chord of 90 Degrees on the Arc, and from the Point where that falls, ſet off the Chord of the Exceſs of the ∠ above 90 Degrees.

Pl. V. Fig. 13. 236. PROBLEM 12. *To draw a Line through a given Point* A, *parallel to a given Line* BC.

CONSTRUCTION, &c. Take any Point D in the Line BC, and join the Points A, D; then make the ∠ EAD [a] = ∠ ADC, and the Line EF will be [b] ‖ to BC. Q. E. I. & Q. E. D.

[a] 230. [b] 73.

Pl. V. F. 14. 237. PROBLEM 13. *At any given Diſtance to draw a Line parallel to another given Line* AB.

CONSTRUCTION. In the Line AB take any Point C, and erect CE [a] ⊥ to AB; then on it ſet off the given Diſtance from C to G, and make the ∠ KGC [b] = ∠ BCG, then will the Line IK be [c] ‖ to AB. Q. E. I. & Q. E. D.

[a] 228. [b] 230. [c] 40.

238.

238. *Scholium.* This and the laſt Problem may be performed ſomething more expeditiouſly, though not ſtrictly geometrical, by the following Methods.

PROBLEM 12th *may be thus conſtructed.* Pl. V. Fig. 15.

Setting one Leg of the Compaſſes in the Point A, open the other ſo as, being turned about, it may touch, but not cut the Line BD (as repreſented by the Arc *ab*;) then take any Point C in the Line BD, and placing one Point of the Compaſſes in it, with the ſame Extent of the Compaſſes, with the other Leg deſcribe the Arc *cd*; then lay a Ruler ſo as to touch the Arc *cd* and the Point A, and draw the Line EA; which will be that required.

The 13th PROBLEM *may be thus conſtructed.*

With a Radius = the given Diſtance, one Leg of the Compaſſes being in C, (a Point taken any where in the Line AB,) with the other deſcribe the Arc *ab*: Again, take another Point D in the Line AB, and and putting one Leg of the Compaſſes in it, with the other Leg and the ſame Radius deſcribe the Arc *cd*; then applying a Ruler to touch theſe two Arcs, draw the Line EF for that required. Theſe two Methods of drawing parallel Lines are grounded on the * Theorem that parallel Lines are every where equidiſtant. Pl. V. Fig. 16.

* 96.

239. PROBLEM 14. *To conſtruct a parallel Ruler for the more ready drawing parallel Lines.*

CONSTRUCTION. Let AB, CD, be two Rulers, very ſtraight, and every where of an equal Breadth, joined together by two Braſs or Silver Pieces, EG, FH, of equal Length, moveable about the Pins E, F, G, H. Note, the Diſtance between the Pins Pl. V. Fig. 17.

EF muſt be = the Diſtance between the Points, G, H; then will the Figure repreſent a parallel Ruler.

DEMONSTRATION. For whatever Diſtance we
open the Rulers AB, CD, to, they will always be
∥ to each other. For, EG being = FG, and EF
= GH, by the Conſtruction, and EH common, the
△s EGH, EFH, have 3 Sides of one reſpectively
= the 3 Sides of the other, and ∵ the △s are equal
a 95. in [a] all Reſpects; conſequently the ∠ GHE = ∠
b 73. FEH, and ∵ AB [b] ∥ to CD. Q. E. D.

240. Figure 18th repreſents another Method of making a parallel Ruler. The Braſs Pieces EH, GF, are of equal Length, joined together exactly in the Middle by a Pin I, about which they are moveable, in the ſame Manner as a Pair of Sciſſars. The Points F and H are moveable about the Pins F, H; and the Ends, E and G, move in Slits made in the Ends, A and C, of the Rulers.

DEMONSTRATION. That the Rulers, AB, CD,
being opened to any Diſtance, will always be ∥ to
each other, will appear by conſidering that FI, HI,
EI, GI, are = each other, by the Conſtruction,
and the ∠ EIF [a] = ∠ GIH, and ∵ the △s EIF, GIH,
a 69. b 58. are [b] equal in all Reſpects; and conſequently the ∠
c 73. FEI = ∠ IHG, and ∵ the Rulers AB, CD [c] ∥ to
each other. Q. E. D.

SCHOLIUM. *The Manner of Uſing a parallel Ruler*, when a Line is to be drawn parallel to a given Line, at a Diſtance not exceeding that to which the Ruler will open, ſeems too evident to need any Deſcription: But when the given Point thro' which the parallel Line is to be drawn, is at a greater Diſtance from the given Line than the Ruler can be opened, in ſuch Caſe it may be convenient to hint, that having placed the Edge of one of the Rulers, ſuppoſe CD, on the given Line, and opened the

Ruler as much as may be towards the given Point, keep the Ruler AB, which is neareſt that Point, firm in its Poſition, and then bring the other Leg CD cloſe to it; where ſtopping it, move forward the Ruler AB, and thus proceed, alternately moving each Ruler carefully, (ſo as to continue always parallel to each other,) until the Ruler AB touches the given Point; when the required Line may be drawn by the Edge of that Ruler.

241. PROBLEM 15. *To triſect an Angle* ACB, *that is, to make an Angle equal to one third of a given Angle.* Pl. V. F. 19.

CONSTRUCTION. About C, as a Center, with any Radius deſcribe the Semicircle DEH, and produce BH at Pleaſure. Then prick a fine Pin in the Point E, and on the Edge of a Ruler GI ſet off GF = the Radius of the Semicircle; then move the Ruler ſo that the End G may always be in the Line BH produced, and the Edge of the Ruler alſo touch the Pin E; and obſerve when the Point F on the Edge of the Ruler touches the Circumference of the Semicircle; then ſtop the Ruler in that Poſition, and draw the right Line GE; then will the $\angle$FCH be $\frac{1}{3}$ of the $\angle$ECD, or, which is the ſame, the Arc HF = $\frac{1}{3}$ of the Arc DE. Q. E. I.

Note. This Problem is impoſſible to be conſtructed by a Method ſtrictly geometrical, either by right Lines, or Circles; which is our Reaſon for giving the above mechanical Method.

DEMONSTRATION. Let the Number of Degrees contained by the $\angle$ FGC be denoted by a; then, FG being = FC by the Conſtruction, the $\angle$ FCG = $\angle$FGC = a Degrees; ∵ $\angle$FCG + $\angle$FGC = $a + a = 2a$. But the $\angle$ EFC being the external $\angle$ of the $\triangle$GFC is[b] = the two internal and oppoſite $\angle$s, ∵ = $2a$; but FC = EC by the Conſtruction, ∵ the $\triangle$FCE is Iſoſceles, and ∵ $\angle$EFC[b] = $\angle$ FEC [a] 59. [b] 76.

= $2a$

[c] 78. $= 2a$ by the above. Now the $\angle$FCE [c] $=$ *180°. *minus* the Sum of the two $\angle$s EFC, and FEC, that is, $= 180^\circ - 4a$. But the Arcs HF, FE, are the Meaſures of the $\angle$s GCF, and FCE, reſpectively; $\because$ the Arc HE$=$FE $+$ Arc HF $= 180^\circ - 4a + a$, that is, $= 180^\circ - 3a$ (for one a being to be added, and $4a$ ſubtracted; it is manifeſtly the ſame if we only ſubtract $3a$.) Now a Semicircle being $=$ 180° the Arc ED muſt be $=$ 180° *minus* Arc HE, that is, $=$ 180° *minus* $\overline{180^\circ - 3a}$, that is, $= 180^\circ - 180^\circ + 3a$, (to ſubtract a Negative being the ſame as to add an Affirmative) $= 3a$; Hence ED $=$ 3 Times HE, or, which is the ſame, the $\angle$FCG $= \frac{1}{3}$ of $\angle$ACB. Q. E. D.

Pl. V. Fig. 20. 242. PROBLEM 16. *To deſcribe a Parallelogram that ſhall be equal to a given Triangle* ABC, *and have one of its Angles equal to a given Angle* D.

[a] 226. CONSTRUCTION. Biſect [a] BC in E, (and join
AE) and at the Point E, in the Line EC, make the
[b] 230. $\angle$ CEF [b] $= \angle$D; then through the Point A draw
[c] 236. AG [c] $\parallel$ to EC, and alſo CG $\parallel$ to EF; then is
EFGC the Parallelogram which was required.

DEMONSTRATION. The $\angle$ ECF being $= \angle$D,
CG $\parallel$ to EF, and FG $\parallel$ to EC, by the Conſtruc-
[d] 97. tion, the Figure EFGC is a [d] Parallelogram, and one
of its $\angle$s CEF $=$ the given $\angle$D. And becauſe BE
[e] 109. $=$ EC, the $\triangle$ AEB [e] $= \triangle$ AEC, and the $\triangle$ AEC
and Parallelogram EG, ſtanding on the ſame Baſe
and between the ſame Parallels, the Parallelogram
[f] 106. is [f] double of the $\triangle$AEC, and conſequently $=$ the
whole $\triangle$ABC. Q. E. D.

Pl. V. Fig. 21. 243. PROBLEM 17. *To apply a Parallelogram to a given Line* AB, *which ſhall be equal to a given Triangle* C, *and have one of its Angles equal to a given Angle* D.

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* By 180° is ſignified 180 Degrees.

Construction. Make the Parallelogram BEFG
= the △C, and having the △ EBG [a] = ∠D, ſo [a] 242.
that BE be in the ſame Line with AB. Then pro-
duce FG to H, and through A draw AH [b] ∥ to [b] 236.
GB or FE, and join the Points H, B, and produce
FE, HB, till they meet each other, and let K be
the Point where they meet: Draw KL ∥ to EA or
FH, and produce GB, HA, to M and L; then
will ABML be the Parallelogram which was re-
quired.

Demonstration. For KL being ∥ to FH,
and FK ∥ HL, by the Conſtruction, the Figure
FKLH is a [c] Parallelogram, whoſe Diameter is HK,
and AG, ME, are Parallelograms about HK, and [c] 97.
LB, BF are the Complements; ∵ BL [d] = BF; [d] 113.
but BF = △C by the Conſtruction, ∵ BL [e] = △C,
and ∠ EBG = D by the Conſtruction, and ∠ [e] 45.
ABM [f] = ∠EBG, ∵ ∠EBG [e] = ∠ ABM; con- [f] 69.
ſequently the Parallelogram BL is = the △C, and
has one ∠ABM = the ∠D. Q. E. D.

244. **Problem** 18: *To deſcribe a Parallelogram* Pl. VI.
equal to a given rectilineal Figure ABCD, *and having* Fig. 1.
an Angle equal to a given rectilineal Angle E.

Construction. Join BD, and make the Pa-
rallelogram FH [a] = the △ ADB, and having the ∠ [a] 242.
HFK = the ∠E; and to the Line GH apply the
Parallelogram GM [b] = the △ BDC, having the ∠ [b] 243.
GHM = ∠E; then will the Figure FLMK be the
Parallelogram required.

Demonstration. For the ∠s FKH, GHM, being each = ∠E, by the Conſtruction, it is manifeſt KM is one Rt. Line, and ∵ the Parallels to it, FG, GL, paſſing through the ſame Point G, will be one Rt. Line FL; and GM being a Parallelogram, LM is ∥ to GH, and, for the ſame Reaſon, FK ∥

to GH, and conſequently LM, FK are Parallels, and ∵ the Figure FLMK a Parallelogram, having the ∠ FKM = ∠ E; and the Parallelogram FH being = △ ADB, and Parallelogram GM = △ BDC, the whole Parallelogram FLMK muſt be = the whole Figure ADCB. Q. E. D.

245. Scholium 1. After the ſame Manner it is manifeſt a Parallelogram may be deſcribed, equal to any given rectilineal Figure, how many ſoever the Sides are; for Example, if the given Figure contained one △ more, it would be only to apply a Parallelogram to the Line LM equal to that third △, and having one ∠ = ∠ E, by Prob. 17.

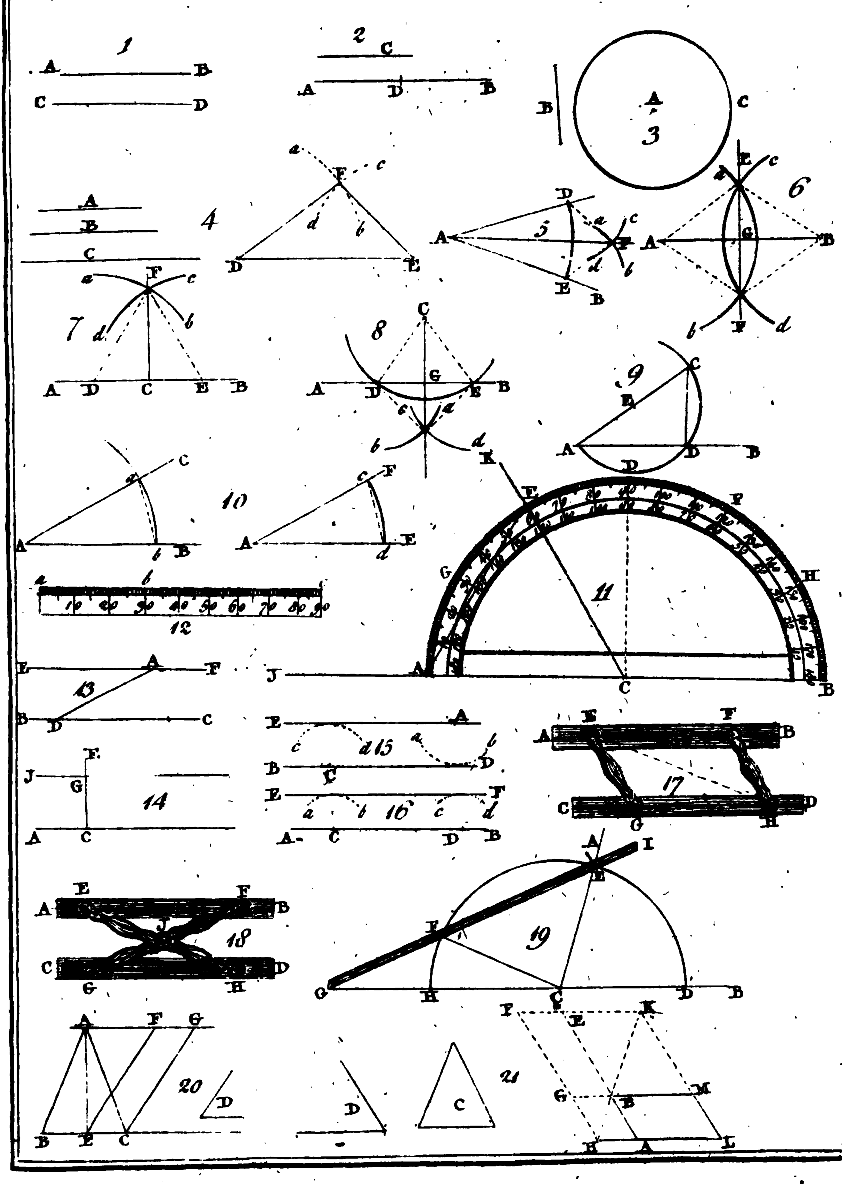
Pl. VI. Fig. 2. 246. Scholium 2. *By this Problem may be eaſily found a Parallelogram equal to the Exceſs of one right-lined Figure given, above another right-lined Figure given.* Thus, if to any Line, CD, a Parallelogram DF is applied = A, and to the ſame Line CD a Parallelogram DH = B, it is manifeſt their Difference, or the Parallelogram GF will be = the Difference between the Figures A and B.

Scholium 3. *Hence alſo it is manifeſt how to find a Parallelogram = the Sum of any two rectilineal Figures.* For if the Parallelogram DH be made = A, and on GH the Parallelogram GF = B, and have an ∠ HGE = ∠ CDG, the Parallelogram DF muſt be = Parallelogram DH + Parallelogram GF, or, which is the ſame, A + B = Parallelogram DF.

Pl. VI. Fig. 3. 247. Problem 19. *To deſcribe a Rectangle whoſe Sides ſhall be equal to two given Lines* AB, EF.

Construction. On one of the given Lines AB, at one Extremity A, erect the Perpendicular AD; then take AB in the Compaſſes, and ſetting one Leg in D, with the other deſcribe the Arc *ab*; after the ſame

P.84. Plate V.



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ſame Manner, with the Extent of EF, and one Leg in B, with the other ſweep the Arc *cd*, and let C be the Point of Interſection; then join DC, BC, and the Thing is done.

DEMONSTRATION. The oppoſite Sides being equal by the Conſtruction, the oppoſite ∠s are alſo[a] equal; and ∵ the ∠ DAB being a Rt. ∠ by the Conſtruction, the other ∠s muſt be ſo too: And ∵ the Figure ABCD is a[b] Parallelogram, and is that required, becauſe its Sides are by the Conſtruction = the given Lines. Q. E. D. [a] 98. [b] 35.

248. SCHOLIUM. *Hence we have an eaſy Method for conſtructing a Square*; for if EF and AB are equal, the Rectangle will be a[c] Square. [c] 34.

249. PROBLEM 20. *To make a Square equal to any Number of given Squares,* e. g. *equal to three given Squares, whoſe Sides are equal to the Lines* A, B, C. Pl. VI. F. 4.

CONSTRUCTION. Draw two indefinite Lines ED, DF, at Rt.[d] ∠s to each other, and take DG = A, and DH = B, and join GH, (and the Line DH will be the Side of a Square = the two Squares whoſe Sides are A and B.) Again, take DI=GH, DK = C, and join KI, then will KI be the Side of a Square = the three given Squares. Q. E. I. [d] 228 or 232.

DEMONSTRATION. For GH^2[e] $= GD^2 + DH^2$; but DG = A, and DH = B, by the Conſtruction, and ∵ $DG^2 = A^2$, and $DH^2 = B^2$, conſequently GH^2[f] $= A^2 + B^2$. Again, $KI^2 =$ [a] $DK^2 + DI^2$; but $DI^2 = (GH^2 =)\ A^2 + B^2$ by the above, and $DK^2 = C^2$, by the Conſtruction, conſequently, $KI^2 =$[b] $A^2 + B^2 + C^2$. Q. E. D. ——— After the ſame Manner may a Square be conſtructed = any Number of given Squares. [e] 115. [f] 45. [a] 115. [b] 45.

250.

Pl. VI. Fig. 5. 250. PROBLEM 21. *To make a Square equal to the Difference of two given Squares, whoſe Sides are* AB *and* CD.

CONSTRUCTION. On one Extremity B, of the ſhorteſt Line AB, raiſe a ⊥ BE, then taking CD in the Compaſſes, put one Leg in A, and with the other deſcribe the Arc *ab*, interſecting BE in F; draw AF, and it will be the Side of the required Square.

c 115. DEMONSTRATION. For $AB^2 + BF^2$ [c] $= AF^2$,
d 47. ∵ taking AB^2 from both Sides, we have BF^2 [d] $=$
$AF^2 - AB^2$; but $AF = CD$ by the Conſtruction,
e 45. ∵ $AF^2 = CD^2$, conſequently BF^2 [e] $= CD^2 - AB^2$.
Q. E. D.

Pl. VI. Fig. 6. 251. PROBLEM 22. *To make a Trianlge equal to any given quadrilateral Figure* ABCD.

CONSTRUCTION. Draw the Diagonal BD, and ∥ to it draw CE interſecting AD produced in E; then join EB, and the △ ABE will be that which was required.

DEMONSTRATION. For the △s DCB, DEB,
being on the ſame Baſe BD, and between the ſame
Parallels, (or, which is the ſame Thing, having
a 109. equal Altitudes,) are [a] = each other; ∵ △ DEB
+ △DBA = △DCB + △DBA; but the △DCB
+ △DBA = the whole quadrilateral Figure ABCD,
and △DEB + △DBA = whole △EBA, ∵ the △
EBA = the Quadrilateral ABCD. Q. E. D.

Pl. VI. Fig. 7. 252. PROBLEM 23. *To make a Triangle equal to any given five-ſided Figure* ABCDE.

CONSTRUCTION. Join EC, AC, and draw DF ∥ to EC, and BG ∥ to CA, meeting AE produced,

duced, in the Points F and G, reſpectively; then draw CF and CG, and the $\triangle$FCG will be that required.——This is ſo very much like the laſt Problem that it needs no Demonſtration.

253. PROBLEM 24. *To deſcribe a Circle through three given Points, (provided they are not in a right Line.)* Pl. VI. Fig. 8.

CONSTRUCTION. Let A, B, C, be the three Points given; draw the Lines AB and BC, and [a] biſect them by the Lines DE, FG, and the Point of their Interſection H will be the Center of the Circle; ∴ put one Leg of the Compaſſes in H, and open the other Leg to one of the given Points, and with that Extent ſweep a Circle, and the Circumference will paſs thro' the three given Points. [a] 226.

DEMONSTRATION. Suppoſe AH, BH, CH, joined; then DB being = DA, and DH common, and the contained $\angle$ ADH = the contained $\angle$ BDH by the Conſtruction, the Side AH [b] = Side BH; and by the ſame Reaſoning BH = CH, ∴ AH = BH = CH; conſequently H is the [c] Center of a Circle whoſe Circumference will paſs thro' the Points A, B, and C. Q. E. D. [b] 52. [c] 136.

254. SCHOLIUM. *By this it is manifeſt, the Center of any given Circle may be found,* by aſſuming any three Points in its Circumference, and proceeding as above. *By this alſo, when we have a Segment of a Circle given, we may compleat the whole Circle.*—— *This Problem alſo ſhews how to deſcribe a Circle about a given Triangle.*

255. PROBLEM 25. *To draw a Tangent to a Circle, from a given Point* D *without it.* Pl. VI. Fig. 9.

CONSTRUCTION. Let B be the Center of the Circle; join BD, which biſect in [a] C; then about C, as [a] 226.

as a Center, with the Radius CB, or its equal CD, deſcribe a Semicircle BAD, and draw DA from D to A, the Point where the Circumference of the Semicircle interſects that of the Circle, and DA will be the Tangent required.

DEMONSTRATION. For if BA be joined, the ∠
b 148. BAD is a [b] Rt. ∠, being in a Semicircle; and ∵
c 141. DA is a [c] Tangent to the Circle at the Point A.
Q. E. D.

256. SCHOLIUM. *If it was required to draw a Tangent from a given Point* A, *in the Circumference of the Circle*, it would be only to erect a ⊥ AD to the Radius BA at the Point A.

Pl. VI. 257. PROBLEM 26. *Upon a given Line* AB, *to*
Fig. 10. *deſcribe a Segment of a Circle, which ſhall contain a given Angle* C.

d 230. CONSTRUCTION. Make the ∠ DAB [d] = ∠C;
e 228. and on AD, at the Point A, erect [e] the ⊥ AI, and
a 226. [a] biſect AB by the Line FH; then let the Point G,
where the Lines AI, FH, interſect, be a Center, about which, with the Radius GA, deſcribe the Circle ABE, and the Segment AEB will be that required.

DEMONSTRATION. Suppoſe B, G, joined; then
AF being = BF by the Conſtruction, FG com-
b 227. mon, and contained ∠AFG [b] = contained ∠BFG,
c 58. the Side AG is [c] = Side BG, ∵ a Circle deſcribed
about G, as a Center, will paſs thro' the Points A
and B; let E be the Point where the Circle inter-
d 151. ſects AI, and join BE, then the ∠AEB [d] = BAD;
but the ∠BAD = ∠C, by the Conſtruction, ∵ the
e 45. ∠AEB [e] = ∠C; that is, the Segment AEB contains
an ∠ = the given ∠C. Q. E. D.

258. Scholium. If the given ∠ is a Rt. ∠, we have only on AB to deſcribe a Semicircle; becauſe the ∠ in a Semicircle is a [f] Rt. ∠. [f] 148.

259. Problem 27. *To cut off a Segment from a given Circle* ABC, *which ſhall contain an Angle equal to a given Angle* D. Pl. VI. Fig. 11.

Construction. To any Point B, in the Circumference, draw [a] the Tangent EF, and make the ∠ FBC [b] = the given ∠ D; then is the Segment BAC that required. [a] 256. [b] 230.

Demonstration. For the ∠ FBC [c] = BAC, and the ∠ FBC alſo = ∠ D, by the Conſtruction, ∴ ∠ BAC [d] = ∠ D. Q. E. D. [c] 151. [d] 45.

260. Problem 28. *To * inſcribe in a given Circle,* BCD, *a Line equal to any given Line,* A, *leſs than the Diameter of the Circle.*

Construction. About any Point B, in the Circumference, as a Center, with a Radius = the given Line A, deſcribe the Arc *ab*, interſecting the Circumference in C; join BC, and the Thing is done. This needs no Demonſtration, it being manifeſt from the Conſtruction that BC = A. Pl. VI. Fig. 12.

261. Problem 29. *In a given Circle* ABC, *to ‡ inſcribe a Triangle equiangular to a given Triangle* DEF. Pl. VI. Fig. 13.

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* A Line is ſaid to be inſcribed, or placed in a Circle, when the Extremities of it are in the Circumference of the Circle.

‡ A rectilineal Figure is ſaid to be inſcribed in a Circle when all the angular Points of the inſcribed Figure are upon the Circumference of the Circle.

Construction. To any Point A, in the Cir-
[a] 256. cumference, [a] draw the Tangent GAH, and make
[a] 230. the ∠ HAC [a] = ∠ DEF, and the ∠ GAB = ∠
DFE, and join BC; then is the △ ABC that required.

[b] 151. Demonstration. For ∠ HAC [b] = ∠ ABC,
and ∠ HAC = ∠DEF by the Conſtruction, ∴ ∠
[c] 45. ABC [c] = ∠ DEF: By the ſame Reaſoning, ∠
ACB = DFE. ∴ the remaining ∠BAC = the re-
maining ∠ EDF, and ſo the △s ABC, DEF, are
[d] 80. equiangular. Q. E. D.

Pl. VI. F. 14. 262. Problem 30. *About a given Circle* ABC, *to* † *deſcribe a Triangle equiangular to a given Triangle* DEF.

Construction. Produce EF both Ways to G
and H; from the Center K draw a Radius KB, and
[a] 230. make the ∠ KBA [a] = ∠ DEF, and ∠ BKC = ∠
DFH, and thro' the Points A, B, C, draw the
Tangents LAM, MBN, NCL, and the △ LMN
will be that required.

Demonstration. The ∠s MAK, MBK,
[a] 142. being [a] Rt. ∠s, and the four ∠s of any quadrilate-
[b] 87. ral Figure being [b] = four Rt. ∠s, the ∠AMB is =
[c] 64. 2 Rt. ∠s − ∠BKA; but ∠DEF [c] = 2 Rt. ∠s —
∠DEG, and ∠ BKA = ∠ DEG by the Conſtruc-
[d] 45. tion, ∴ ∠AMB [d] = 2 Rt. ∠s — ∠DEG; ∴ ∠
[e] 45. AMB [e] = ∠DEF; and by the ſame Reaſoning,
∠BNC = ∠DFE. Hence we have ſhewn that the
two ∠s LMN, LNM, are reſpectively = the two
∠s DEF, DFE, and, conſequently, the remaining
[f] 80. ∠MLN [f] = the remaining ∠EDF; and ∴ the ∠s
LMN, DEF, are equiangular. Q. E. D.

263.

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† A rectilineal Figure is ſaid to be deſcribed about a Circle when each Side of the circumſcribing Figure touches the Circumference of the Circle.

263. Problem 31. *To* * *inſcribe a Circle in a* Pl. VI.
given Triangle ABC. Fig. 15.

Construction. Biſect [g] the ∠s ABC, BCA, [g] 225.
by the Lines BI, CH, and let D be the Point of In-
terſection, and draw DF [h] ⊥ to BC; then about D, [h] 229.
as a Center, with a Radius = DF, deſcribe a Circle
FGE, and it will be that required.

Demonstration. Let DE be ⊥ to BA, and
DG ⊥ to AC; then the ∠s BED, BFD, being
[a] Rt. ∠s, are = each other; and the ∠ EBD [a] 14.
= ∠ FBD by the Conſtruction, and the Side BD
common, ∵ the △s EBD, FBD, are = [b] each [b] 94.
other in all Reſpects, and conſequently ED = DF.
By the ſame Reaſon DG = DF; ∵ ED, DF, DG,
are = each other, and conſequently the Circle deſcribed about the Center D with the Radius DF,
will paſs thro' the Points FEG, or, in other Words,
will touch the Sides of the given Triangle in the
Points E, F, and G. Q. E. D.

264. Problem 32. *To inſcribe a Square in a* Pl. VII.
given Circle ABCD. F. 1.

Construction. Through the Center E draw
the Diameter AC, and [c] ⊥ to it the Diameter AC, [c] 226.
and join BA, AD, DC, BC, then is the rectilineal
Figure ABCD the Square required.

Demonstration. BE being = ED, AE com-
mon, and the ∠ AEB [d] = ∠ AED, the Side AB [d] 14.
will be [e] = Side AD; and, by the ſame Reaſon, AB [e] 58.
= BC, and AD = DC; ∵ AB [a] = BC = AD = [a] 45.
DC; that is, the rectilineal Figure ABCD is equi-
lateral. And BD being a Diameter, the ∠ BAD is

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* A Circle is ſaid to be inſcribed in a rectilineal Figure when the Circumference touches each Side of the rectilineal Figure.

[b] 148. an ∠ in a Semicircle, and ∴ a Rt. [b] ∠; By the
ſame Reaſon the ∠s ABC, BCD, CDA, are each
of them Rt. ∠s. Hence we have ſhewn that the
rectilineal Figure ABCD is equilateral, and its ∠s
[c] 34. all Right ∠s. Conſequently it is a [c] Square. Q. E. D.

Pl. VII. F. 2. 265. Problem 33. *To deſcribe a Square about a given Circle* ABCD.

Construction. Draw the Diameters AC, BD,
[d] 226. [d] at Rt. ∠s to each other, and thro' the Points A,
[e] 256. B, C, D, draw the [e] Tangents GF, GH, HK,
KF; then will the Figure FGHK be the required
Square.

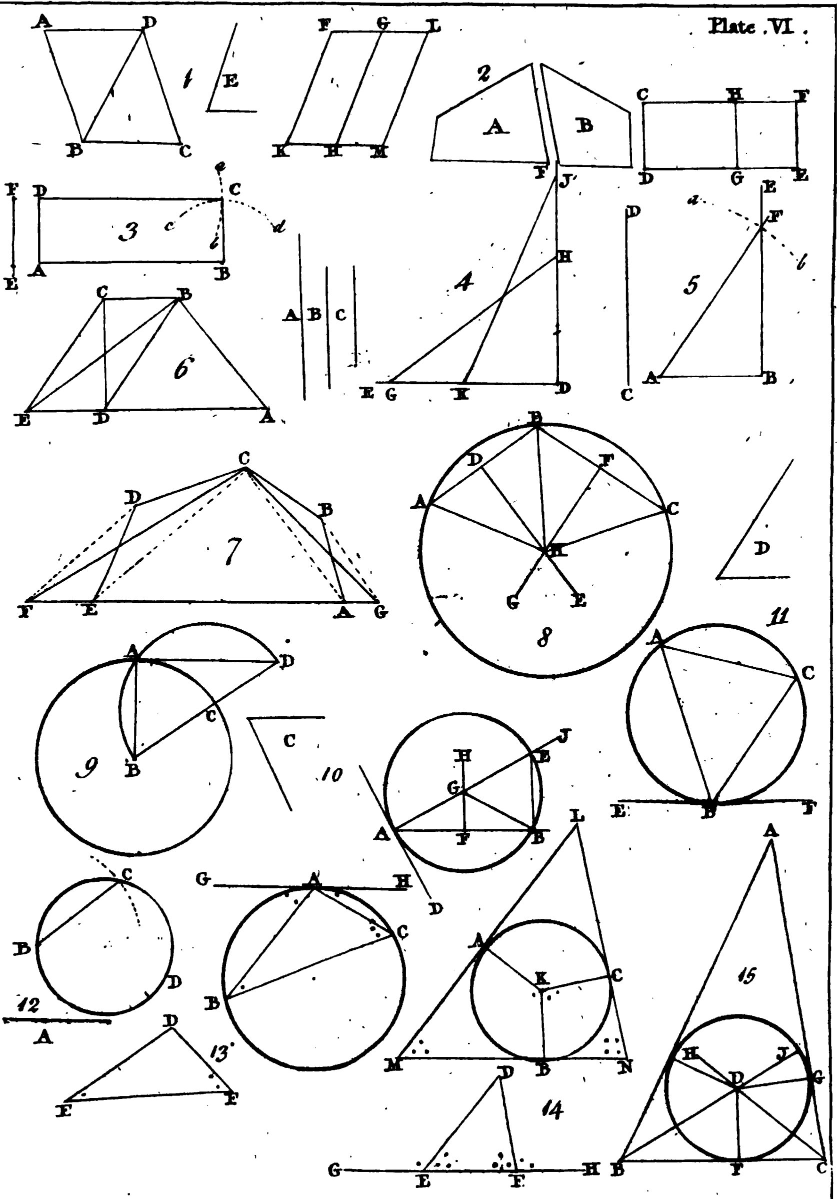
Demonstration. For the ∠s at A, B, C, D,
are right ∠s by the Conſtruction, and becauſe the
∠s GBE, AED, are each of them Rt. ∠s, they
[f] 40. are equal to each other, and ∴ GH [f] ∥ to AC;
[g] 74. and by the ſame Reaſon FK ∥ to AC, ∴ GH [g] ∥
to FK. After the ſame Manner it may be ſhewn
that GF is ∥ to HK; hence the Figure FGHK is
[a] 97. a [a] Parallelogram, and its oppoſite Sides and ∠s are
equal; alſo GC, AK, FB, BK, are Parallelograms,
and ∴ their oppoſite Sides are equal; hence GF =
BD in the Parallelogram FB, and GH = AC in the
Parallelogram GC; but AC = BD being each the
Diameter of the Circle, ∴ GF = GH; and the
oppoſite Sides of the Figure FGHK having been
ſhewn above to be equal, it follows that GF = GH
= HK = FK, or that it is equilateral: It is alſo
rectangular, for the ∠s GAE, AEB, EBG, being
[b] 87. all Rt. ∠s, the ∠AGB muſt be alſo a [b] Rt. ∠: After
the ſame Manner it may be ſhewn that the ∠s
at the Points H, K, F, are Rt. ∠s; ∴ the Figure
GFKH has been proved to be both equilateral
[c] 34. and rectangular; ∴ it is a [c] Square. Q. E. D.

Pl. VII. F. 2. 266. Problem 34. *To inſcribe a Circle in a given Square* FGHK.

Con-

P. 92.

Plate . VI .



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Construction. [d] Biſect the Sides GF, GH, [d] 226.
by the Lines BD, AC; then about the Point of
Interſection E, with the Radius EA, ſweep the
Circle ABCD, and it will be that which was re-
quired.

Demonstration. The ∠s AGB, EBH, being
both Rt. ∠s by the Conſtruction, are = each other,
and ∴ BE [a] || to GA; by the ſame Reaſon AE || [a] 40.
to GB; ∴ the Figure AGBE is a [b] Parallelogram, [b] 97.
and ∴ its oppoſite Sides are equal, *viz.* AE =
GB, and BE = GA: But GB, GA, being the
Halves of equal Lines, *viz.* of GF, GH, (by the
Conſtruction) are [c] = each other, ∴ AE = GB = [c] 51.
BE = GA, or the Figure is equilateral; it is alſo
rectangular, for the ∠s EAG, AGB, GBA, being
Rt. ∠s by the Conſtruction, the other ∠ AEB muſt
be a [d] Rt. ∠ alſo; and ∴ the quadrangular Fi- [d] 87.
gure AGBE a [e] Square. After the ſame Manner [e] 34.
AFDE, DKCE, CHBE, are ſhewn to be Squares
= the former; ∴ EA = EB = EC = ED, and
conſequently, a Circle deſcribed about the Center E,
with the Radius EA, will paſs thro' the Points A,
B, C, D, and [f] touch the Sides of the given Square [f] 141.
in thoſe Points, (as the ∠s at thoſe Points are Rt. ∠s.
Q. E. D.

267. Problem 35. *To deſcribe a Circle about a* Pl. VII.
given Square ABCD. F. 1.

Construction. Draw the Diagonals AC, BD; then, about the Point of Interſection E, as a Center, with the Radius EA, deſcribe a Circle, and it will be that required.

Demonstration. AB being = DC, AD com-
mon, and ∠BAD = ∠ADC, BD [a] = AC; and [a] 58.
EA [b] = EC, and ED [b] = EB. But the whole Dia- [b] 14.
gonals AC, BD, have been juſt ſhewn to be equal,

∴

∵ their Halves muſt be equal alſo, that is, EA = ED = EC = EB; and conſequently a Circle deſcribed about the Point E, with a Radius = EA, muſt paſs thro' the Points A, B, C, and D. Q. E. D.

268. Before we proceed any further, it may be proper to give ſome Definitions concerning *Polygons.*

A *Polygon* is generally defined to be a Plane Figure terminated by more than four Sides. ——— According to its Etymology it is a Figure having many Angles; which is in Fact the ſame as the above Definition; for when a Figure has many Sides it muſt have many Angles, *et contrà*; the Number of Sides and Angles being always equal. — When it is both *equilateral* and *equiangular*, it is called *regular*, otherwiſe *irregular.*

A Polygon of	5	Sides is called a	Pentagon
	6		Hexagon
	7		Heptagon
	8		Octagon
	9		Enneagon
	10		Decagon
	11		Endecagon
	12		Dodecagon
	&c.		&c.

Note. In what follows we would be underſtood to treat of regular Polygons only.

269. The Angle made by any two contiguous Sides of a Polygon is called the *Angle of the Polygon.* And the Angle made by two Lines drawn from the Center of the Polygon, to the extreme Points of the ſame Side of the Polygon, is called the *Angle at the Center.*

270. A LEMMA: *Lines biſecting the Angles of the Polygon, will interſect each other in one Point within the Polygon, and may be called Radii, for they will be equal to each other.* Pl. VII. Fig. 3.

For the ∠s of the Polygon being [a] equal, their Halves muſt be ſo too: ∵ the ∠ABS = ∠BAS, ∵ SA [b] = SB; and by the ſame Reaſon SB = SC, &c. ∵ AS = SB = SC, &c. Q. E. D. [a] 268. [b] 62.

271. COROLLARY 1. *Hence, Lines drawn from the Center of a Polygon to the Extremities of each Side, will divide the Polygon into as many Iſoſceles Triangles, equal to each other in all Reſpects, as the Figure has Sides.*

272. COROLLARY 2. *Hence alſo, each Angle at the Center* (as all the Angles at that Point are [c] = 4 Rt. ∠s, or 360 Degrees,) *muſt be* = 360 *Degrees divided by the Number of Sides.* [c] 68.

273. COROLLARY 3. *Hence alſo it follows, that the Angle of the Polygon is* = 180° − *the* ∠ *at the Center.*

For ∠ ABS + ∠SAB [a] = 180° − ∠ at the Center; but ∠ ABS = ∠ SAB by the [b] above, ∵ (∠ ABS + ∠ABS, or, which is the ſame) 2∠ABS = 180° − ∠ at the Center; but 2∠ABS is = ∠ of the Polygon, ∵ the ∠ of the Polygon = 180° − ∠ at the Center. [a] 78. [b] 270.

274. From the two laſt Corollaries the following Table is eaſily conſtructed.

Name

Name of the Polygon.	Angle at Center.	Angle of the Polygon.
Pentagon	72° 00′	108° 00′
Hexagon	60 00	120 00
Heptagon	51 $25\frac{5}{7}$	128 $34\frac{4}{7}$
Octagon	45 00	135 00
Enneagon	40 00	140 00
Decagon	36 00	144 00
Endecagon	32 $43\frac{7}{11}$	147 $16\frac{4}{11}$
Dodecagon	30 00	150 00

Pl. VII. Fig. 3. 275. PROBLEM 36. *In a given Circle to inſcribe a Polygon of a given Number of Sides.*

CONSTRUCTION. Draw a Radius AS, and make the ∠ ASB = the ∠ at the Center; draw AB, and it will be one of the Sides of the Polygon. Take AB in the Compaſſes, and ſet off that Diſtance from B to C, from C to D, &c. then will the Figure ABC, &c. be the required Polygon. This is manifeſt.

276. COROLLARY. *In a Hexagon, the Side of the Polygon is equal to the Radius*, viz. AB = AS; for in that Caſe the ∠s ASB, ABS, SAB, are each = 60°, and ∵ the △ ASB equilateral. *Note*, we ſhall defer ſaying more on the Conſtruction of Polygons till we treat of the Conſtruction and Uſe of an excellent Inſtrument called a *Sector*.

Pl. VII. Fig. 4. 277. PROBLEM 37. *From a given Line*, AB, *to cut off any Part required; for Example, a Fifth of* AB.

CONSTRUCTION. Draw an indefinite Line AC, making any ∠ with AB; on which ſet off any Diſtance A1, ſet off the ſame from 1 to 2, from 2 to 3, from 3 to 4, from 4 to C; then have we ſet off as many equal Parts on AC as AB is ſuppoſed to be divided into. Join B5, and || to it, draw 1*a*, then will A*a* be $\frac{1}{5}$ Part of AB.

DEMONSTRATION. BC, *a*1, being || to each other, as C1 : A1 [a] :: B*a* : A*a*, ∵ compoundedly C1 + A1 : A1 [b] :: B*a* + A*a* : A*a*, that is, CA : A1 :: BA : A*a*; conſequently, A*a* is the ſame Part of AB as A1 is of CA. Q. E. D.

[a] 191.

[b] 200.

278. SCHOLIUM. By this Problem a Line AB is divided into any Number of equal Parts. For Example Sake, A*a* being $\frac{1}{5}$ Part of AB, if we ſet the Diſtance A*a* from *a* to *b*, from *b* to *c*, and from *c* to *d*, the ſame would be = *d*B, and ſo the Line AB divided into five equal Parts. And after this Manner Scales may be conſtructed. But for this Purpoſe, *viz. to divide a given Line into any Number of equal Parts*, the following way is better adapted. For Example Sake, let it be required to divide the Line AB (ſee Plate VII. Fig. 20.) into five equal Parts.

CONSTRUCTION, From the Point A draw the indefinite Line AC, making any Angle at Pleaſure with the given Line AB; and || to AC, draw the indefinite Line BD: Then taking any convenient Diſtance in your Compaſſes, ſet it from A to 1, from 1 to 2, 2 to 3, &c. alſo ſet the ſame Diſtance from B to *a*, *a* to *b*, *b* to *c*, &c. then the Line being to be divided into 5 equal Parts, join B5, and you cannot miſtake in joining the other correſpondent Points, *a*4, *b*3, *c*2, *d*1; and theſe Lines will divide the given Line AB into 5 equal Parts, *viz.* A*f* = *fg* = *gh* = *hi* = *i*B.

DEMONSTRATION. It is manifeſt from the Conſtruction, that the Lines *a*4, *b*3, &c. are all of them || to B5, and to each other, and therefore the △s A1*f*, A2*g*, &c. are ſimilar to each other; ∵ as

$A1 : Af :: A2 : Ag$; but if we denote $A1$ by m, and Af by n, $A2$ will be $= 2m$, and ſo the Analogy become $m : n :: 2m : Ag$; but it is manifeſt Ag muſt be $= 2n$, for $m : n$ has the ſame Ratio as $2m : 2n$; $\because$ $Ag = 2n$, and conſequently $fg = n$. Again, as $A1 : Af :: A3 : Ah$, that is, $m : n :: 3m : Ah$, conſequently, Ah muſt be $= 3n$, to preſerve the ſame Ratio: But Ag has been juſt ſhewn to be $= 2n$, $\because$ gh ($= Ah - Ag$ muſt be) $= n$ alſo. After the ſame Manner hi and iB may be proved to be equal to n, and ſo Af, fg, hi, iB = one and the ſame Quantity, and conſequently equal to each other; or, in other Words, the Line AB is divided into five equal Parts, in the Points f, g, h, i, and B. Q. E. D.

N. B. We ſhall treat of the Conſtruction and Uſe of diagonal Scales in our Eſſay on Trigonometry.

279. PROBLEM 38. *To find a third Proportional to two given Lines.*

Pl. VII. Fig. 5. CONSTRUCTION. Let AB, AC, be the two given Lines, ſo placed as to contain any Angle. Produce AC, AB, at Pleaſure; in AB produced take BD = AC; join BC, and thro' D draw DE || to it; then will CE be the third Proportional required.

[a] 191. DEMONSTRATION. For as AB : AC[a] :: BD : CE; but BD = AC by the Conſtruction, $\because$ it is, as AB : AC :: AC : CE. Q. E. D.

Pl. VII. Fig. 6. 280. PROBLEM 39. *To find a fourth Proportional to three given Lines* A, B, C.

CONSTRUCTION. Draw two indefinite Lines, Da, Db, making any Angle. In Db take DG = A, and GE = B; and in Da take DH = C; join GH, and draw EF ∥ to it; then will HF be the fourth Proportional. Q. E. I.

Demonstration. For as DG : GE [a] :: DH : HF; or, which is the ſame, A : B :: C : HF. Q. E. D. [a] 191.

281. Problem 40. *To find a mean Proportional to two given Lines,* a, b. Pl. VII. Fig. 7.

Construction. Draw an indefinite Line A*d*, in which take BD = *a*, DC = *b*, and upon BC deſcribe a Semicircle; and on the Point D raiſe the ⊥ DA, to cut the Circumference of the Semicircle in A, and it will be the mean Proportional required.

Demonstration. Join BA, CA, then it is manifeſt from Art. 210, that DA is a mean Proportional between BD and DC, or, which is the ſame, between the Lines *a* and *b*. Q. E. D.

N. B. This ſhews alſo how to make a Square = a given Rectangle. For if BD, DC, be the Sides of the Rectangle, DA is the Side of a Square = it: For by Art. 211, BD × DC = DA^2.

282. Problem 41. *To divide a Line* BC *according to extreme and mean Proportion.* Pl. VII. Fig. 8.

Construction. Draw BE ⊥ to BC and = $\frac{1}{2}$ BC; and about E as a Center, with the Radius EB, deſcribe a Circle, and thro' the Points C, E, draw the Line CA, cutting the Circumference of the Circle in the Points D and A; then make CF = CD, and the given Line BC will be divided by the Point F, into extreme and mean Proportion, as was required.

Demonstration. By Art. 213, CA × CD = BC^2; ∵ BC : CA [a] :: CD : BC, whence inverſe- [a] 194.

[b] 198. ly, CA : BC [b] :: BC : CD ; this dividedly is
[c] 201. CA — BC : BC [c] :: BC — CD : CD ; which in-
[d] 198. verſely is, BC : CA — BC [d] :: CD : BC — CD. By the Conſtruction CF = CD, and DA, being a Diameter, is = twice BE, or = BC, by the Conſtruction ; ∵ CA = CF + BC ; hence, by writing CF for its equal CD, and CF + BC for its equal CA, in the laſt Analogy, we have BC : CF + BC — BC :: CF : BC — CF = BF, or, which is the ſame, (as + BC — BC deſtroy each other,) BC : CF :: CF : BF. Q. E. D. (See Art. 176.)

283. PROBLEM 42. *To make a Square equal to any given right-line Figure.*

CONSTRUCTION. By Problem 18, find a Rectangle = the given Figure ; then, by Problem 40, find a Square = that Rectangle.

284. SCHOLIUM. When the given Figure has not more than five Sides, perhaps the eaſieſt Way to find a Rectangle = the given Figure, is, firſt to find, by Problem 22, or 23, a △ = that Figure ; and then a Rectangle = that △ may be found by Prob. 16.

285. PROBLEM 43. *To find two Lines which ſhall have the ſame Ratio to each other as any two given rectilineal Figures.*

CONSTRUCTION. Take two Lines at Pleaſure = each other, and on them make Rectangles = the reſpective given Figures ; then are the Rectangles
[a] 190. in the ſame [a] Ratio as their Baſes, (their Altitudes being equal by the Conſtruction ;) conſequently, the given Figures muſt be in the ſame Ratio as the Baſes of the Rectangles.

286,

286. PROBLEM 44: *Upon a given Line* FG *to deſcribe a rectilineal Figure, ſimilar, and ſimilarly ſituated, to a given rectilineal Figure* ABCDE. Pl. VII. Fig. 9.

CONSTRUCTION. Join BE, BD, and make the ∠GFK = ∠A, and ∠FGK = ∠ABE; then make ∠GKI = ∠BED, and ∠KGI = ∠ EBD. Again, make ∠ GIH = ∠BDC, and ∠ IGH = ∠ DBC, and the Figure FGHIK will be that required. After the ſame Manner, by proceeding from △ to △, a Figure may be made ſimilar to another of any Number of Sides.

DEMONSTRATION. The ∠ GFK being = ∠ BAE, and ∠ FGK = ∠ ABE by the Conſtruction, the remaining ∠ FKG muſt be [a] = the remaining ∠AEB, and ∵ the △s GFK, BAE, equiangular, and conſequently [b] ſimilar: After the ſame Manner the △ GKI is ſhewn to be ſimilar to the △ BED, and the △GIH ſimilar to the △BDC; conſequently the Polygons FGHIK, ABCDE, which are compoſed of the ſimilar △s ſimilarly ſituated, muſt be ſimilar to each other. Q. E. D.

[a] 80.

[b] 194.

287. SCHOLIUM. When we treat of *Surveying*, we intend to give ſome practical Methods for ſolving this Problem in a more expeditious Manner.

BOOK

BOOK VI.

Of the Application of Geometry to ſome practical Problems for meaſuring Heights, with the Addition of ſome pleaſant Problems and Paradoxes for the Amuſement of young Minds.

Pl. VII. Fig. 10. 288. Prob. 1. *TO find the Height of an acceſſible Object by Means of two Poles, of an unequal Length.*

Solution. Let AB repreſent the ⊥ Object, whoſe Height is to be found, above the Plane DB. —— Let ED, the ſhorteſt Pole, be erected ⊥ at any Point D, (which may be done by a Plumb Line, ſuch as Maſons commonly uſe in building Walls upright,) then, fixing your Eye at the upper End of it E, let another Perſon carry along the longeſt Pole, till you ſee its Top appear in a direct Line with A, the Top of the Object, where let it be ſtuck down; then meaſure the Diſtance DC, and CB. It is evident the △s EFG, EAH, are ſimilar, and ∴ we may ſtate by the Golden Rule, as EG (or DC) : FG (the Difference of the Lengths of the two Poles) :: EH (or DB) : HA. And AH + HB (the Height of the ſhorteſt Pole) gives BA the Height of the Object. Q. E. I.

Pl. VII. Fig. 11. 289. A Lemma. *If a Ray of Light,* EC, *falls on a reflecting Surface (as a Piece of Looking-Glaſs)* C, *and is reflected to the Eye at* A, *the Angles* ACB, ECD, *are equal to each other.*

This is demonſtrated by the Writers on Optics, but muſt here be taken for granted, till we treat of that Science.

290. **Problem** 2. *To find the Height of an acceſſible Object by Means of any plain reflecting Surface, ſuch as a Looking Glaſs, or Bowl of Water.*

Solution. Let ED repreſent the Height of the Object ⊥ to the Ground BD; at any Place on the Ground C, place the reflecting Surface; then go directly backward in the Line DC produced to B, till E the Top of the Object is ſeen by Reflection in the reflecting Surface at C, by the Eye at A; then meaſure the Height of the Eye BA, the Diſtances BC, and CD. Then the ∠ ACB being = ∠ ECD by the Lemma, and the Angles ABC, CDE, each Rt. ∠s, the remaining ∠ BAC = the remaining ∠ DEC, and ſo the △s ABC, EDC, equiangular, and ∴ ſimilar. Hence we may ſtate by the Golden Rule, as CB : BA :: CD : DE, the Height of the Object required.

N. B. In our *Eſſay on Decimal Arithmetick*, Art. 60, is given a Method of finding the Height of an Object by its Shadow.——— We ſhall treat more fully, and illuſtrate Methods more accurate in Practice, after we have treated of the Elements of Trigonometry.

291. **Problem** 3. *To deſcribe a Figure called an Oval.* Pl. VII. Fig. 12.

Construction. Draw a Line AB, about B, with any Radius BA ſweep a Circle AECK, with the ſame Extent of the Compaſſes about C, as a Center, ſweep a Circle BHDI; let E and F be the Points where the Circles interſect each other; join F, B, and produce FB to G; in like Manner draw FH, EK, and EI; then ſetting one Leg of the Compaſſes in F, with the Radius FG deſcribe the Arc GH; laſtly, one Point of the Compaſſes being placed in E, with the other deſcribe the Arc KI; then is the Figure AGHDIK called an *Oval.*

Pl. VII. Fig. 13. 292. Problem 4. *To deſcribe a Figure called by ſome an Helix, or Spiral Line.*

Construction. Draw a Line AB, and about any Point *a*, with any Radius *ac*, ſweep the Semicircle *ced*; then take the Point *b* ſo that *bc* = *ba*, and about it, as a Center, with the Radius *bd*, deſcribe the Semicircle *dgf*; then about the Center *a*, with the Radius *af*, deſcribe a Semicircle *fmn*, and thus proceed at Pleaſure, making the Points *a* and *b* alternately the Center of the Semicircle to be deſcribed.

Pl. VII. F.14.&15. 293. Problem 5. Paradox 1. *Having a round Table, to make it into a ſquare Table, and have a Hole the Middle, without Waſte.*

Solution. Let the round Table be cut, as repreſented by the croſs Diameters in Figure 14, then place the four Parts together, as ſhewn in Figure 15, and the Thing is done.

294. Problem 6. Paradox 2. *Having a round Table, to change it into two Oval-like Tables, and have a Hole in the Middle of each, without Waſte of Timber.*

Pl. VII. F.16&17. Solution. Let the Table be cut into eight Pieces, as repreſented in Figure 16, then with the four Parts, 1, 2, 3, 4, may be made an Oval-like Table, as ſhewn in Figure 17; and with the other four Parts may be made another ſimilar and equal to Figure 17. Q. E. I.

295. Problem 7. Paradox 3. *A Gentleman having a Hole exactly a Foot ſquare, to try the Skill of his Joiner, has ordered him to fill up that Hole with a Piece*

Piece of Board in Length 16 Inches, and Breadth 9 Inches, with a ſtrict Charge to cut the Piece into two Parts only: How may the Joiner perform his Taſk?

Solution. Let the Board, Figure 18, be cut as repreſented by the ſtrong Lines *ab*, *bc*, *cd*, *de*, *ef*; then put the two Parts A*abcdef*B and *a*CD*fedcb* together, as repreſented in Figure 19, and the Thing is done. Pl. VII. F. 18. & 19.

296. Problem 8, Paradox 4. *To deſcribe Circles of different Radii, with the ſame Extent of the Compaſſes.* Pl. VIII. Fig. 1.

Construction. With any Extent of the Compaſſes, and one Leg in A, with the other deſcribe a Circle: then, taking a little Solid, (ſuppoſe a Die,) reſting one Leg of the Compaſſes on it as a Center B, with the other deſcribe a Circle on the Paper, and it is manifeſt it will be of leſs Radius than the other.

297. Problem 9, Paradox 5. *To deſcribe an oval-like Figure at one Sweep of the Compaſſes.*

Construction. Take a round Solid, (ſuppoſe a round Ruler,) and on it roll a Piece of Paper; then, having opened the Compaſſes to any convenient Extent, fixing one Leg at any Point aſſumed on the Ruler, turn the other round as if you were to deſcribe a Circle: Then take off the Paper, and the Figure ſo deſcribed will repreſent a Kind of Oval, being longer one Way than the other.

298. Problem 10, Paradox 6. *To cut a Hole in a Card, large enough for a Man to paſs through.*

Solution. Let ABCD repreſent the Card. In, or near the Middle, cut a Slit, as repreſented by the Line *ma*, each End of it almoſt cloſe to the Edge Pl. VIII. Fig. 2.

of the Card; then cut, as ſhewn by the Line *a*1; again, from 2 to 3, 4 to 5, 6 to 7; and thus proceed until you have cut the Line *mn*, and one Half of the Card is finiſhed. This being done, and the other Part cut in the ſame Manner, on opening the Card, you will find a Hole ſufficiently large for a Man to paſs through. *N. B.* The nearer the Lines *a*1, 2, 3, &c. are to each other, the larger will the Hole be.

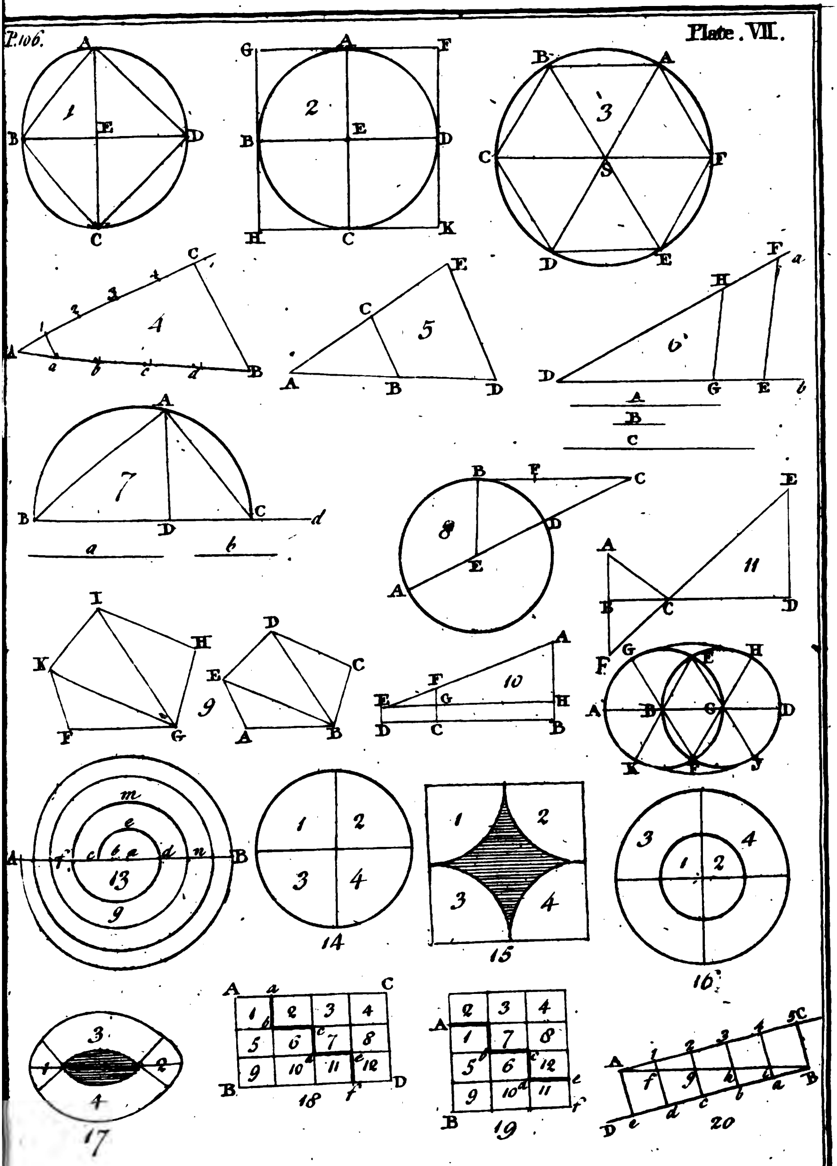
299. PROBLEM II. *To deſcribe an Ellipſis of any given Length and Breadth.*

Pl. VIII. Fig. 3. CONSTRUCTION. Draw the Line AC = the given Length, and biſect it by the Perpendicular DB, making ED, EB, each = $\frac{1}{2}$ the given Breadth; then is AC called the *Tranſverſe Axis*, and BD the *Conjugate Axis*. Take EA or EC in the Compaſſes, and ſetting one Foot in B, with the other deſcribe two Arcs interſecting the Tranſverſe in the Points F and G; which Points are called the *Foci*. Then take two Pins, and placing one in the Point A, and the other in C, join them by a ſilk or other convenient String: This done, remove the Pins from the Points A and C to the Foci F and G; then take a Pencil in your Hand, and with it keep the String always extended, as repreſented at H; then if you begin at A and carry your Hand from A towards D, and from thence to C, the Point of the Pencil will deſcribe the Half of the Curve ADC: And in the ſame Manner, beginning again at A, proceed towards B, and from thence to C, and you will have deſcribed the other Half ABC; and the Figure ABCD ſo deſcribed is called an Ellipſis.

Otherwiſe without a String, thus.

Open the Compaſſes to any Diſtance leſs than the tranſverſe Axis AC, and with it about the Focus F, as a Center, deſcribe the Arcs *ab*, *ef*; and with the ſame Extent of the Compaſſes about the Focus G, as a Center, the Arcs *ik*, *no*. This being done, ſet the

Plate . VII.



UNIV

the Diſtance (to which the Compaſſes are opened) from A towards C on the tranſverſe Axis, *viz.* to I: Then, taking the remaining Part of the Tranſverſe IC in the Compaſſes, about G as a Center, deſcribe the Arcs *cd*, *gh*: and with the ſame Radius, about F as a Center, deſcribe the Arcs *lm*, *pq*; then will the Points of their Interſections *r*, *v*, *s*, *t*, be 4 Points in the Curve of the Ellipſis. After the ſame Manner, by taking different Radii at Pleaſure, as many Points may be found in the Curve as may be ſufficient to join by an even Hand, to form the required Ellipſis.

SCHOLIUM. This Figure is one of the Sections of a Cone, and therefore 'is intended to be more particularly treated of hereafter; but as it is in daily Uſe amongſt Joiners, Coopers, and other Mechanics, very few of whom can be ſuppoſed to ſtudy *Conic Sections*, we thought it would be of Service to ſome of them, to give the moſt eaſy Way of deſcribing it in this Place.

300. PROBLEM 12. *To deſcribe a Figure reſembling a Hen's Egg.*

CONSTRUCTION. From a Scale of equal Parts ſet off 32 from A to G, and 43 from G to H. Make the ∠AGF and ∠DHC each = 40°. Produce FG until GI is = 21 (on the ſame Scale of equal Parts.) Then about G as a Center, with the Radius GA deſcribe the Arc AF; and about I as a Center, with the Radius IF, deſcribe an Arc FE until the Chord FE is = the Radius IF. Again, through the Point I draw the Line EK to interſect DH produced, in L: Then, about L as a Center, with the Radius LE deſcribe the Arc ED. Laſtly, about H as a Center, with the Radius HD deſcribe the Arc DC, and the Curve AFEDC will repreſent Half of the Figure required. The other Half may be thus deſcribed, *viz.* About G as a Center de- Pl. VIII. Fig. 4.

ſcribe

ſcribe the Arc AM = AF; and about H as a Center, the Arc CP = CD. Produce MG, and make MN = FI; then about N as a Center, deſcribe the Arc MB until its Chord MB is = the Radius NM. Produce BN, and make BO = EL; then about O as a Center deſcribe the Arc BP; and the Figure ABCEA will be a very natural Repreſentation of a Hen's Egg.

N. B. If the Numbers are ſet off from a Scale of 40 or 45 to an Inch, (in the Plate annexed we have uſed that of 45,) the Figure will be about the Size of a real Egg, that from which we deduced this Conſtruction being 42 to an Inch.

Scholium. Many more Problems might have been given in this Chapter, but as they are intended chiefly for the Amuſement of young Minds, perhaps theſe which have been already given, may by ſome Perſons be thought more than ſufficient.

A N

AN
APPENDIX,
CONTAINING
Some ADDITIONAL NOTES, *a* GEOMETRICAL SPECULUM, *and an* INDEX *to the whole Volume.*

301. A *Note* on the 6th, 7th, 8th, 9th, 10th, 11th, and 12th Articles.

In theſe Articles we have given the Definitions in the Order in which they ſtand in *Euclid*, becauſe we ſet out with a Reſolution to follow him as nearly as we conveniently could, conſiſtent with our Plan; though at the ſame Time we cannot but acknowledge, that as they have all their Exiſtence in a Solid, the moſt natural Way ſeemed to be, firſt to define a Solid, and then to ſhew how they exiſt in it; which will be extremely natural and eaſy, and take away the Objections commonly made againſt *Euclid*'s Definitions, by ſuch Perſons as are not uſed to Abſtraction: And ſince this Eſſay was ſent to the Preſs, we find Profeſſor *Simſon* has moſt elegantly illuſtrated this, in the 2d. Edition of his *Euclid*, and therefore, as we cannot give a better, take the Illuſtration in his own Words, *viz.*

“It is neceſſary to conſider a SOLID, *that is a Magnitude which has Length, Breadth, and Thickneſs*, in Order to underſtand aright the Definitions of a Point, Line, and Superficies; for theſe all ariſe from a Solid and exiſt in it. *The Boundary, or Boundaries, which contain a Solid, are called* SUPERFICIES, *or the Boundary which is common to two Solids which are contiguous or which divides one Solid into two contiguous Parts, is called a* SUPERFICIES. Thus, if BCGF be one of the Boundaries which contain the Solid ABCDEFGH, or which is the common Boundary of this Solid and the Solid BKLCENMG, and is therefore in the one as well as the other Solid, it is called a Superficies, and has no Thickneſs. For if it have any, this Thickneſs muſt either be a Part of the Solid AG, or of the Solid BM, or a Part of the Thickneſs of each of them. It cannot be a Part of the Thickneſs of the Solid BM, becauſe if this Solid be removed from the Solid AG, the Superficies BCGF, the Boundary of the Solid AG, remains ſtill the ſame as it was. Nor can it be a Part of the

Pl. VIII. Fig. 5.

Thickneſs of the Solid AG, becauſe if this be removed from the Solid BM, the Superficies BCGF, the Boundary of the Solid BM, does nevertheleſs remain. Therefore the Superficies BCGF has no Thickneſs, but only Length and Breadth.

The Boundary of a Superficies is called a LINE, *or a Line is the common Boundary of two Superficies that are contiguous, or which divides one Superficies into two contiguous Parts.* Thus, if BC be one of the Boundaries which contain the Superficies ABCD, or which is the common Boundary of this Superficies and of the Superficies KBCL, which is contiguous to it, this Boundary BC is called a Line, and has no Breadth. For if it have any, this muſt be Part either of the Breadth of the Superficies ABCD, or of the Superficies KBCL, or Part of each of them. It is not Part of the Breadth of the Superficies KBCL, for if this Superficies be removed from the Superficies ABCD, the Line BC, which is the Boundary of the Superficies ABCD, remains the ſame as it was. Nor can the Breadth that BC is ſuppoſed to have be a Part of the Breadth of the Superficies ABCD, becauſe if this be removed from the Superficies KBCL, the Line BC, which is the Boundary of the Superficies KBCL, does nevertheleſs remain. Therefore the Line BC has no Breadth. And becauſe the Line BC is in a Superficies, and that Superficies has no Thickneſs, as was ſhewn; therefore a Line has neither Breadth or Thickneſs, but only Length.

The Boundary of a Line is called a POINT, *or a Point is the common Boundary or Extremity of two Lines that are contiguous.* Thus, if B be the Extremity of the Line AB, or the common Extremity of the two Lines AB, KB, this Extremity is called a Point, and has no Length. For if it have any, this Length muſt either be Part of the Length of the Line AB, or of the Line KB. It is not Part of the Length of KB, for if the Line KB be removed from AB, the Point B, which is the Extremity of the Line AB, remains the ſame as it was: Nor is it Part of the Length of the Line AB; for if AB be removed from the Line KB, the Point B, which is the Extremity of the Line KB, does nevertheleſs remain. Therefore the Point B has no Length. And becauſe a Point is in a Line, and a Line has neither Breadth nor Thickneſs, therefore a Point has no Length, Breadth, nor Thickneſs. And in this Manner the Definitions of a Point, Line, and Superficies are to be underſtood."

302. The 8th Definition, which we have omitted, is not only *uſeleſs*, but *obſcure*; and as Mr. *Simſon* juſtly obſerves, ſeems to be the Addition of ſome leſs ſkilful Editor.

303. (Art. 57.) Having by Experience found that in printing the common Characters for *greater* and *leſs*, they are frequently turned the wrong Way, and ſo when it ſhould be *greater* have *leſs*, and the contrary. In Order to remove the Confuſion occaſioned hereby, we have introduced the Greek G (Γ) for *greater than*, and the Greek L (Λ) for *leſs than*; which entirely prevent ſuch Accidents, and at the ſame Time carry with them the Ideas

we have Occaſion to expreſs, and therefore are no Burden to the Memory.

304. (Art. 94.) The laying of one Figure on another has been objected to as a mechanical Conſideration, depending on no Poſtulate: But for our Parts we muſt acknowledge that we ſee no Reaſon why this Method of Demonſtration ſhould not be uſed, as its Evidence cannot be denied by any Perſon: And as there is no Occaſion for the actual placing of one Figure on the other, but only to conceive them in the Mind to be ſo placed, it does not appear to us to be more mechanical than the conceiving of a Line to be drawn, or a Circle deſcribed. And *Euclid* and all other Geometricians make Uſe of it in demonſtrating our 58th Article, which indeed does not ſeem capable of any other Method. As to its being grounded on no Poſtulate, that Objection may be eaſily removed, by making a fifth Poſtulate (after Article 44.) viz. *Grant that one Figure may be laid on another*; and certainly no Perſon who grants the others, will deny this, it being as manifeſt as either of them. Thus much we thought ourſelves obliged to remark, as the Objection was made by a late learned Geometrician, for whoſe Memory we have the greateſt Eſteem.

305. (Art. 115.) As this Theorem is one of the moſt valuable in Geometry, it may perhaps be expected that we ſhould give an Example or two of its Uſe. By this it is that any two Sides of a right-angled Triangle being given, we find the other Side: This admits of two Caſes.

Case 1. *The two ſhort Sides being given, to find the longeſt.*

From the *Theorem* this *Rule* is eaſily deduced, *viz.* * The Square Root of the Sum of the Squares of the two ſhort Sides will be equal to the longeſt Side.

Example. Let DC repreſent Half the Breadth of a Houſe, and be = 15 Feet, and CE the Height of the Roof = 10 Feet; it is required to find the Length of the Rafter DE.

Pl. VIII. Fig. 10.

The Solution by the above Rule is as follows.

15	10	225	325(18
15	10	100	1
75	$CE^2 = 100$	$DC^2 = 325$	28)125
15			224
$DE^2 = 225$			

Hence the Length of the Rafter is ſomething more than 18 Feet.

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* Demonstration. Let DC be expreſſed by b, CE by p, and DE by h, then by the Theorem, $b^2 + p^2 = h^2$ ∴ extracting the Square Root of each Side of the Equation, we get $\sqrt{b^2 + p^2} = h$. Q. E. D.

After the ſame Manner the Side of a Square being given, its Diagonal may be found nearly true: We ſay nearly, becauſe it does not admit of an exact Solution; for if we ſuppoſe the Side = 1, the Sum of the Squares of both Sides will be = 2, the Square Root of which is 1.41 nearly, but cannot be found quite accurate, and therefore, though a finite Quantity, cannot be accurately expreſſed in Numbers. Such Quantities are called *incommenſurate*.

Case 2. *When the longeſt Side and one of the ſhort Sides are given, to find the other we have this* Rule. From the Square of the longeſt Side ſubtract the Square of the given ſhort Side, and the Square Root of the Remainder will be the ſhort Side required.*

But by Article 121. this may be worded ſo as to be more ready in Practice, *viz.*

Multiply the Sum of the two given Sides by their Difference, and extract the Square Root of their Product.

Example. A Ladder 50 Feet long reached the Top of a Houſe when its Foot was 30 Feet from the Bottom of the Wall; it is required to find the Height of the Houſe.

The Operation will ſtand thus, by the firſt Rule.

$$\begin{array}{r} 50 \\ 50 \\ \hline 2500 \\ \hline \end{array} \qquad \begin{array}{r} 30 \\ 30 \\ \hline 900 \\ \hline \end{array} \qquad \begin{array}{r} 2500 \\ 900 \\ \hline 1600 \\ \hline \end{array} \qquad \begin{array}{l} \dot{1}6\dot{0}0(40 \\ 16 \\ \hline \end{array}$$

By the other Rule thus.

$$\begin{array}{rr} & 50 \\ & 30 \\ \hline \text{Sum} & 80 \\ \hline \end{array} \qquad \begin{array}{r} 80 \\ 20 \\ \hline 1600 \\ \hline \end{array} \qquad \begin{array}{l} \dot{1}6\dot{0}0(40 \\ 16 \\ \hline \end{array}$$

Difference 20

Anſwer, the Houſe is 40 Feet high.

306. (Book 2. Art. 117.) This Definition is according to *Euclid*, but every Perſon knows, that *two* right Lines cannot contain a Space, that is, cannot incloſe any Thing, as it at leaſt requires *three* right Lines for that Purpoſe; and therefore this muſt not be taken in a ſtrict Senſe: But when a Rectangle or right-angled Parallelogram is ſaid to be contained by any two Lines, *Euclid* muſt only be underſtood to denote a Rectangle made on one of the Lines, and having an Altitude equal to the other.

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* Demonstration, By the Theorem $b^2+p^2=h^2$, ∵ taking p^2 from both Sides of the Equation, we have $b^2=h^2-p^2$ ∴ by extracting the Square Root we have $b=\sqrt{h^2-p^2}$. Q. E. D.

307. (Book 3. Art. 144.) This Remark is alſo ſupported by Mr. *Simſon* (who is undoubtedly one of rhe beſt Judges in theſe Matters) for in Page 301 of the ſecond Edition of his *Euclid* he has this Note, *viz.* "In this we have not followed " the *Greek*, nor the *Latin* Tranſlation literally, but have given " what is plainly the Meaning of the Propoſition, without mentioning the Angle of the Semicircle, or that which ſome call " the cornicular Angle, which they conceive to be made by " the Circumference and the ſtraight Line which is at right " Angles to the Diameter, at its Extremity; which Angles have " furniſhed Matter of great Debate between ſome of the modern " Geometers, and given Occaſion of deducing ſtrange Conſequences from them, which are quite avoided by the Manner " in which we have expreſſed the Propoſition. In like Manner " we have given the true Meaning of Prop. 31. Book 3. without mentioning the Angles of the greater or leſſer Segments. " Theſe Paſſages *Vieta* with good Reaſon ſuſpects to be adulterated, in the 386th Page of his *Oper. Math.*"

308. (Book 4, Definition 1.) We have here to the Word *Part*, as it is in *Euclid*, prefixed *aliquot*, as it ſeems to be wanting to expreſs *Euclid*'s Meaning; for a leſſer Magnitude may be Part of a greater, and yet not meaſure the greater; *e.g.* the Side of a Square is leſs than the Diagonal, and therefore from the Diagonal may be taken a Part = the Side; but this will not meaſure the Diagonal, being incommenſurate to each other by Article 305.

309. We have before obſerved in Scholium to Article 208, that *Euclid*'s Method of treating the Doctrine of Proportion is too obſcure for young Students, at their firſt Beginning to read theſe Subjects; and conſequently this is a ſufficient Reaſon (as we write for Learners) for attempting to demonſtrate the principal Theorems in another Manner. But as ſome may perhaps expect a Proof of our Aſſertion, that it is too difficult, it may be proper to obſerve, that this evidently appears from the great Diſputes which have been concerning the Principles on which the *Euclidian* Method is grounded amongſt the greateſt Geometricians: For if they are of ſo difficult a Nature that the learned themſelves cannot agree about them, it muſt follow they cannot be proper for initiating Youth, as Things cannot be well too plain for them. Mr. *Thomas Simpſon*, one of the beſt, as well as one of the lateſt Writers on Geometry, thus expreſſes himſelf on this Subject in Page 274, "That there is ſomething very ingenious " and ſubtle in the Doctrine of Proportions as deliver'd in *Euclid*'s 5th Book, cannot be denied. All that I contend for is, " that the Principles on which it is built are *obſcure*, and not ſo " firmly eſtabliſhed as to authorize its Partiſans to aſſume that " great Superiority they lay Claim to, in Point of geometrical " Strictneſs." In Page 270 he thus remarks on the Definition which we have omitted (ſee our 158 Art.) *viz.* "I would ſeriouſly aſk the Contemners of *the vulgar and confuſed Notion* of

“ Proportions, if a Definition, by which it cannot be known whether the Ratio of the firſt to the ſecond of four given Magnitudes may not at the ſame Time be both greater and leſs than that of the third to the fourth, is really calculated to afford thoſe very accurate Ideas they pretend to?” And he even goes ſo far as to affirm that “ it ſeems ſufficiently plain from *Euclid*’s own Authority, that he himſelf was not entirely pleaſed with his own Performance on this Head; or that he was convinced (at leaſt) that it had not every Advantage: For otherwiſe it will be very difficult to account for his having demonſtrated many Things in his 7th Book, by another Method, whoſe Demonſtrations had been actually given before, in the 5th, under a different Form. For theſe Reaſons, when I ſee the extravagant Commendations that have been laviſhed on this 5th Book of *Euclid*, I am no farther convinced by them than that great Men may ſometimes launch out too far in Behalf of Opinions which they have adopted.”

Thus much ſeemed neceſſary for us to extract in our own Vindication; but as we are determined not to enter into any Diſpute on this Subject, we ſhall proceed no further on this Head.

310. We ſhall conclude our Obſervations on this Book with the Theorem hinted at in the 116th Article, viz. *In any Triangle*, ABC, *(the Angle at the Vertex being the greateſt Angle; or at leaſt* =
Pl. VIII. *either of the Angles of the Triangle) make the Angle* ACD $= \angle$B,
F.6,7,&8. *and* $\angle$BCE $= \angle$A; *then will the Sum of the Squares on the two Sides containing the vertical Angle* C, *be equal to the Sum of the Rectangles contained by the whole Side ſubtending that Angle and Segments*, AD, BE, that is, $AC^2 + BC^2 = AD \times AB + BE \times AB$.

N. B. Fig. 6. repreſents the Caſe when the $\angle$C is obtuſe, Fig. 7, when acute, but greater than either of the others, that is, when it is acute, but greater than 60 Degrees. Fig. 8. when the Angle is a right Angle. Fig. 9. when that Angle is not greater, but equal to either of the other Angles, *viz.* = $\frac{1}{3}$ of 2 Right Angles, or 60 Degrees. Theſe Figures ſhew all the Varieties poſſible, and the Truth of the Theorem in all theſe Varieties will appear from one general DEMONSTRATION, *viz.* The $\triangle$s ABC, ACD, BCE, are ſimilar by the Conſtruction. $\because$ Comparing the Triangles ACD, ABC, together, we have AC : AD
* 193. :: AB : AC, $\because AC^2 =$ * $AD \times AB$. Again, by the ſimilar $\triangle$s BCE, ABC, we have BC : BE :: AB : BC $\because BC^2 = BE \times AB$. $\because$ by Addition of Equals, $AC^2 + BC^2 = AD \times AB + BE \times AB$. Q. E. D.

311. COROLLARY 1. When the Angle C is a Rt. $\angle$, the
Fig. 8. Points D and E will coincide (as in Fig. 8.) and hence if on AB we make a Square, it is manifeſt that $AD \times AB + BE \times AB =$ the Rectangle AG + the Rectangle BG, or, which is the ſame, = the Square on AB, and $\because AC^2 + BC^2 = AB^2$; which is the famous *Pythagorean* Theorem in our 115 Article.

312. COROLLARY 2. When the $\angle$ C is = 60°, the $\triangle$ is
Fig. 9. known to be equiangular and equilateral, and the Points A and E, B and D, coincide; and ſo $AD \times AB + BE \times AB$ is the ſame

ſame as $AB^2 + AB^2$, or twice the Square on AB. But this is alſo known to be true from other Principles; for as the three Sides are equal, it is manifeſt their Squares muſt be ſo too, and conſequently the Sum of any two of them muſt be = twice the other.

313. Corollary 3. From the firſt Corollary it evidently appears, that "*If a Line be divided into any two Parts, the Rectangles contained by the whole and each of the Parts are together = the Square of the Whole,* viz. $AG + BG = AH$, or $AB \times AD + AB \times BD = AB^2$. This is the 2d. of *Euclid's* ſecond Book. Fig. 8.

314. Corollary 4 Hence alſo, *if there be two right Lines,* AB, AF, *and one of them,* AB, *is divided into two or more Parts, the Rectangle contained by the two Lines is = the Sum of the Rectangles contained by the undivided Line* AF, *and the ſeveral Parts of the divided Line* AB. Thus it is manifeſt from the Figure that the Rectangle AG + the Rectangle BG = Rectangle AH, or, which is the ſame, $AD \times AF + DB \times$ (BH, or its Equal) $AF = AB \times AF$.

This is the 1ſt. of *Euclid's* ſecond Book.

315. Book 5. Art. 228. It ſometimes happens in Practice, that the Point C is ſo very near B, the End of the Line, that a ſufficient Diſtance cannot be ſet from C towards B, nor the Line by ſome Circumſtance admit of being produced longer; as for Inſtance, the Point C may be almoſt cloſe to the Edge of the Paper; and conſequently in ſuch Caſe *Euclid's* Method is not eaſy in Practice: Therefore it may be neceſſary to give other Methods of Conſtruction peculiarly adapted to this Problem. *To draw a Perpendicular to a Line* AB *from any Point* C, *either in or very near the End* B *of that Line, ſo as to be impracticable by the* Euclidian *Method.* Pl. VIII. Fig. 10.

This may be done ſeveral Ways.

1ſt. From a Scale of equal Parts take off 3, and ſet it from C to D; then from the ſame Scale take off 4, and placing one Point of the Compaſſes in C, with the other deſcribe the Arc *ab*: Again, take off 5, and one Point of the Compaſſes being in D, with the other deſcribe the Arc *cd*, and through the Point of Interſection E draw a Line CF, and it will be the Perpendicular required. For if D, E, be joined, the △ DCE will be a Right ∠△, right-angled at C, by Art. 305.

N. B. Any other Numbers having the ſame Ratio to each other, will do the ſame Thing; as for Example, 6, 8, and 10; or 30, 40, and 50, &c.

Or thus.

Above the Line AB aſſume any Point D at Pleaſure, and about it as a Center, with the Radius DC, deſcribe a Circle (or a ſufficient Part of it,) and through the Point of Interſection E, and Center D, draw the Diameter EF; then through the Points C, F, draw the Line CG, and it will be the Perpendicular required. For the ∠ ECF in a Semicircle is a Right Angle by Art. 148. Pl. VIII. Fig. 11.

There are many other Methods, but theſe are ſufficient.

316. Before we conclude our Notes on this Book, it may be of Uſe to ſome Perſons *to ſhew one of the eaſieſt and beſt Methods of trying whether a Carpenter's Square is truly made*; *viz.* apply the Square to the ſtraight Edge of a Plank AB, as repreſented by Pl. VIII. Fig. 12. the ſhaded Square, and by it draw the Line CD; then taking it off, turn it about, and apply it as ſhewn by the unſhaded Square, and if, when brought to the Point C, it agrees with the Line CD before drawn, the Square is true, if not, the Difference will be double the Error of the Square.

317. Book VI. Art. 289. It is a Maxim made Uſe of by ſome modern Philoſophers, that *Nature*, or rather the God of *Nature*, does every Thing in the *ſhorteſt Way*, and indeed it is a Principle which ſeems to follow from the very Idea of every Thing being ordered by *Infinite Wiſdom*, and as far as it has been applied, has always been found agreeable to the Nature of Things. From this Principle, by a fluxionary Proceſs, the late Writers on Optics ſhew *that the Angle of Incidence muſt be equal to the Angle of Reflection*, which was the Reaſon of our ſaying, "that it muſt be taken for granted until we treat of that "Science:" But recollecting that in reaſoning on *Hadley*'s Octant in 1752, we thought on a very eaſy geometrical Way of demonſtrating it, we ſhall here give it, with which we ſhall conclude our Notes.

Demonstration. Produce AB to F, making $BF = BA$, and join CF. By the Suppoſition $EC + CA$ is to be a *Minimum*. By the Conſtruction $BF = BA$, the $\angle$s ABC, FBC, each Rt. $\angle$s, and BC common, and therefore by Art. 58. the $\triangle$s are equal in all Reſpects, $\because$ $FC = CA$. Hence, when $EC + CA$ is a *Minimum*, its Equal $EC + FC$ muſt be ſo too; but it is evident that $EC + CF$ is a *Minimum* when EF is a Right Line, the neareſt Diſtance to any two Points being a right Line. But when EF is a Rt. Line, the $\angle ECD = \angle FCB$ by Art. 69. but the $\triangle$s ABC, FBC, have been ſhewn to be equal in all Reſpects, $\because$ $\angle FCB = \angle ACB$; $\because$ by Axiom 1, Art. 45, $\angle ECD = ACB$. Q. E. D.

Scholium. Hence it appears that Plane Geometry has in ſome Caſes the Advantage over Fluxions, as this is very conciſe and evident, but the fluxionary Proceſs ſomething tedious and affected with Surds.

As in this Eſſay, for Conveniency of Demonſtration, the Theorems (as in *Euclid*) relating to the ſame Subjects are ſome in one Place, ſome in another; and as they are in continual Uſe in almoſt every Branch of the Mathematics, we ſhall here collect them together in a ſhort View, or as it may be properly called, *A* Geometrical Speculum, which will be found of the greateſt Uſe to all young Beginners.

APPENDIX.

N. B. The Numbers prefixed refer to the Articles in which they are demonſtrated.

I. *Of the Properties of Lines interſecting or making Angles with each other.*

THEOREM 1.

64. A Line ſtanding on another Line making Angles with it, the Sum of thoſe Angles is equal to two right Angles.

2.

65. If one of the Angles (in the laſt Article) be a right one, the other is alſo a right Angle; but if one be acute, the other is obtuſe.

3.

66. If ſeveral Lines ſtand on the ſame Line at the ſame Point, the Angles which they make are together equal to two right Angles.

4.

67. Two Lines croſſing each other make four Angles, which taken together are equal to four right Angles.

5.

68. All the Angles made about a Point are = 4 Rt. ∠s.

6.

69. If two Lines interſect each other, the oppoſite ∠s are equal.

7.

289 If a Ray of Light, falling on a reflecting Surface, is reflected to the Eye, the ∠ of Incidence is = the Angle of Reflection. See Art. 317.

II. *Of Parallel Lines.*

1.

72. If a Line croſſes two Parallel Lines, the alternate ∠s are = each other.

2.

73. If a Line, falling upon two other Lines, makes the alternate ∠s equal to each other, the Lines will be parallel.

3.

74. If 2 Lines be each of them parallel to another Line, they are parallel to each other.

4.

75. If a Line is perpendicular to another Line, it is alſo perpendicular to all the ∥s of that other.

5.

96. Two ∥ Lines, though infinitely continued, could never meet.

III. *Of Rectangles of Lines, or Products of Quantities.*

1.

119. If a Line be divided into any two Parts, the Square on the whole Line is = the Squares on the two Parts + twice the Rectangle contained by the Parts.

2.

120. The Square on any Line is = 4 Times the Square on Half that Line.

3.

121. A Rectangle made by the Sum and Difference of two Lines is = the Difference of the Squares on the ſaid Lines.

4.

313. If a Line be divided into any two Parts, the Rectangles contained by the Whole and each of the Parts are together = the Square of the Whole, *viz.* Pl. VIII. Fig. 8. $AB \times AD + AB \times BD = AB^2$.

5.

314. If there be two Rt. Lines, AB, AF, and one of them, AB, is divided into two or more Parts, the Rectangle contained by the two Lines is = the Sum of the Rectangles contained by the whole Line AF and the ſeveral Parts of the divided Line AB. Thus, $AD \times AF + BD \times AF = AB \times AF$.

IV. *Of the Properties of* TRIANGLES *with Reſpect to their Angles and Sides.*

1.

59. The ∠s at the Baſe of an Iſoſceles △ and oppoſite to the equal Sides are = each other; and a Line biſecting the Angle oppoſite the Baſe divides the Baſe into two equal Parts, and is perpendicular thereto.

2.

60. Every equilateral △ is alſo equiangular.

3.

62. If 2 ∠s of a △ be equal, the Sides ſubtending the equal Angles will be equal.

4.

63. Every equiangular △ is alſo equilateral.

5.

70. Any two Sides of a △ are together greater than the third Side.

6.

76. If any Side of a △ be produced, the external ∠ is = the 2 internal oppoſite ∠s.

7.

77. The external ∠ is Γ either of the internal oppoſite ∠s.

8.

8.

78. The 3 interior ∠s of every △ are = 2 Rt. ∠s.

9.

79. The 3 ∠s of every △ taken together are = the 3 ∠s of any other △ taken together.

10.

80. If in one △ 2 of its ∠s taken together are = 2 ∠s of another △ taken together, the remaining ∠ of one will be = the remaining ∠ of the other.

11.

81. If 2 △s have one ∠ of one = one ∠ of the other, the Sum of the remaining ∠s of one = the Sum of the 2 remaining ∠s of the other.

12.

82. If in any △ one of its ∠s is a Rt. ∠, the Sum of the other 2 is a Rt. ∠.

13.

83. If in a Rt. ∠ △ one of the acute ∠s is given, the other is = the Difference between the given ∠ and a Rt. ∠.

14.

84. When the Angle contained between the equal Sides of an Iſoſceles △ is a Rt. ∠, the other 2 ∠s are each half a Rt. ∠.

15.

85. An equilateral △ hath each ∠ = $\frac{1}{3}$ of 2 Right ∠s = 60 Degrees.

16.

86. If in any △ one of its ∠s is either a Rt. or an obtuſe ∠, the other two will be each of them acute.

17.

89. The greater Side of every △ is oppoſite to the greater ∠.

18.

90. In every △ the greater ∠ is oppoſite to the greater Side.

19.

91. If two △s have the ſame common Baſe, the 2 Sides of the inſcribed △ taken together are leſs than the two Sides of the circumſcribing △; but the vertical ∠ of the inſcribed greater than the vertical ∠ of the circumſcribing △.

20.

92. If 2 △s have 2 Sides of one = 2 Sides of the other, each to each, but the contained ∠ of one Γ the contained ∠ of the other, that which has the greateſt contained ∠, will have the greateſt Baſe.

21.

93. If 2 △s have 2 Sides of one = 2 Sides of the other, each to each, but the Baſe of one Γ the Baſe of the other; the ∠ contained by the equal Sides of that which has the greateſt Baſe ſhall be Γ the ∠ contained by the equal Sides of the other.

22.

96. In any Right ∠ △ the Square upon the Side ſubtending the Rt. ∠ is = both the Squares upon the Sides containing the Rt. ∠.

See another Demonſtration in Note to Art. 211.

23.

310. In any △ ABC (Pl. VIII. Fig. 6, 7, 8, and 9,) (the ∠ at the Vertex being the greateſt, or at leaſt = either of the Angles of the △) make the $\angle ACD = \angle B$, and $\angle BCE = \angle A$; then will the Sum of the Squares on the two Sides containing the vertical ∠ C, be = the Sum of the Rectangles contained by that whole Side ſubtending that ∠, and Segments AD, BE, that is, $AC^2 + BC^2 = AD \times AB + BE \times AB$.

N. B. The Theorems relating to the letting fall a Perpendicular from the vertical Angle to the Baſe of any Triangle, being chiefly of Uſe in Algebra, are referred to our intended Eſſay on that Art.

V. *Of the Equality of △'s in every Reſpect; viz. when all their Angles and Sides are reſpectively equal to each other.*

1.

58. If 2 △s have 2 Sides of the one = the 2 Sides of the other, each to each, and the ∠s contained by thoſe equal Sides equal, the △s will be equal in all Reſpects.

2.

94. If 2 △s have two ∠s of one = 2 ∠s of the other, each to each, and one Side of one equal to one Side of the other, the △s are equal in all Reſpects.

3.

95. If 2 △s have the 3 Sides of one = the 3 Sides of the other, each to each, the △s will be equal in all Reſpects.

VI. *Of the Similarity of Triangles and Proportionality of their Sides, &c.*

1.

191. If a Line be drawn ∥ to one Side of a △, it ſhall cut the other Sides proportional. And if the Sides, or Sides produced, are cut proportionally, the Line which joins the Points of Section ſhall be parallel to the remaining Side of the △.

2.

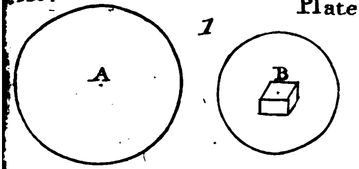
192. If the vertical ∠ of a △ be divided into 2 equal ∠s by a Line cutting the Baſe, the Segments of the Baſe will have the ſame Ratio that the other two Sides of the △ have to one another. And if the Segments of the Baſe have the ſame Ratio which the other Sides of the △ have to one another, the Line drawn from the Vertex to the Point of cutting will divide the vertical Angle into 2 equal Parts.

3.

194. Equiangular △s are ſimilar, that is, the Sides about the equal ∠s are proportional.

Plate VIII.

1



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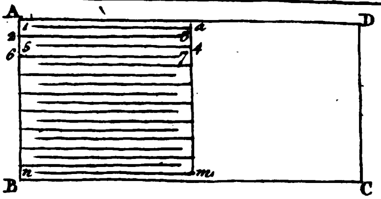

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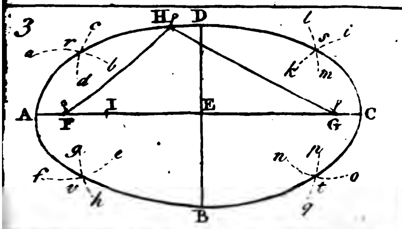

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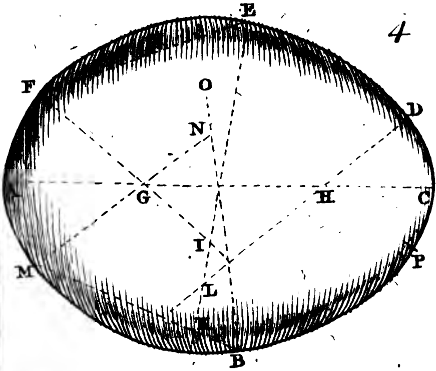

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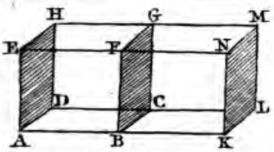

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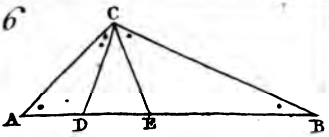

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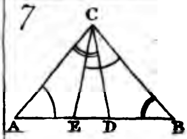

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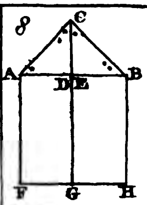

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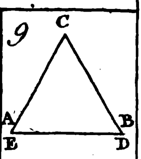

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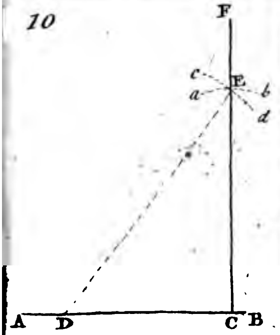

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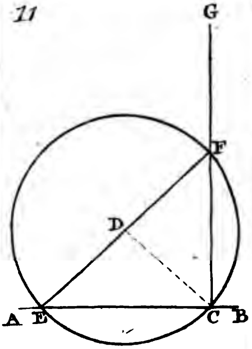

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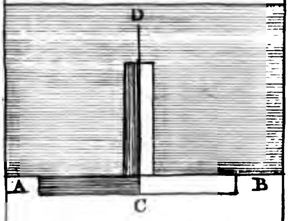

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4.

195. If the outward ∠ of a △ is biſected by a Line cutting the Baſe produced, then the Segments between the dividing Line and the Extremities of the Baſe have the ſame Ratio which the other Sides of the △ have to one another. And if the Segments of the Baſe produced have the ſame Ratio which the other Sides of the △ have, the Line drawn from the Vertex to the Point of Section divides the outward ∠ of the △ into 2 equal Parts.

5.

196. If 2 △s have one ∠ of one = one ∠ of the other, and the Sides containing the ∠s proportional, the △s are equiangular and ſimilar.

6.

197. If 2 △s have their like Sides proportional, the △s are ſimilar to each other.

7.

198. If in a Rt. ∠ △ a Perpendicular be let fall from the Rt. ∠ to the oppoſite Side, the △s on each Side of it ſhall be ſimilar to the whole △ and to each other.

VII. *Of Parallelograms, and the Compariſon of △s and Parallelograms, with Reſpect to their Areas, or Equality of Surface.*

1.

97. The oppoſite Sides and ∠s of Parallelograms are = each other: and the Diameter (or Diagonal) biſects them; that is, divides them into 2 equal Parts.

2.

99. Figures which are between the ſame Parallels have equal Altitudes.

3.

100. Rectangles contained under equal Sides are equal.

4.

101. If 2 Squares have the Side of one = the Side of the other; the Squares are = each other.

5.

102. Parallelograms upon the ſame Baſe, and between the ſame ||s are equal.

6.

103. Parallelograms having the ſame Baſe and equal Altitudes are equal.

7.

104. Every Parallelogram is = a Rectangle having the ſame Baſe and equal Altitudes.

8.

105. Parallelograms having equal Baſes and equal Altitudes are equal.

9.

106. If a △ and Parallelogram ſtand upon the ſame Baſe and are between the ſame Parallels, the △ is $\frac{1}{2}$ of the Parallelogram.

10.

107. If a Parallelogram and △ have equal Baſes and are between the ſame ||s, or, which is the ſame Thing, have equal Altitudes, the △ is half of the Parallelogram.

11.

108. If a △ and Rectangle have equal Baſes and Altitudes, the △ is Half of the Rectangle.

12.

109. If 2 △s have equal Baſes and Altitudes, they are equal to each other.

13.

111. The Area cf any Parallelogram is = the Product of the Number of Parts into which the Baſe is ſuppoſed to be divided by the Number of like Parts contained in its Altitude.

14.

113. The Complements of Parallelograms which are about the Diagonal of any Parallelogram are = each other.

15.

114. The Diagonals of any Parallelogram biſect each other.

16.

189. Triangles and Parallelograms of the ſame Altitude are to one another as their Baſes.

17.

190. Triangles and Parallelograms which have equal Altitudes are to one another as their Baſes.

VIII. *Of Polygons in general.*

1.

87. All the interior ∠s of any rectilineal Figure are together with 4 Rt. ∠s = twice as many Rt. ∠s as the Figure has Sides.

2.

88. All the exterior ∠s of any rectilineal Figure are together = 4 Rt. ∠.

3.

98. All quadrilateral Figures, whoſe oppoſite Sides are equal, have their oppoſite ∠s equal.

4.

114. The Diagonals of any Parallelogram biſect each other.

IX. *Of regular Polygons.*

1.

270. Lines biſecting the Angles of the Polygon will interſect each other in one Point within the Polygon, and may be called Radii, for they will be equal to each other.

2.

271. Lines drawn from the Center of a Polygon to the Extremities of each Side will divide the Polygon into as many Iſoſceles △s, equal to each other in all Reſpects, as the Figure has Sides.

3.

272. Each ∠ at the Center is = 360 Degrees divided by the Number of Sides.

4.

273. The ∠ of the Polygon is = 180° — the ∠ at the Center. See a Table of ∠s in Art. 274.

5.

276. In a Hexagon the Side of the Polygon is = the Radius.

X. *Of the Properties of Circles.*

1.

134. If a Line drawn through the Center of a Circle biſect any Chord, it will cut it at Rt. ∠s: And if the Diameter is at Rt. ∠s to the Chord, it will biſect it.

2.

*134. If any Point which is not the Center be taken in the Diameter of a Circle, then of all the Lines that can be drawn from that Point to the Circumference, that in which the Center is, is the longeſt, and the other Part of the Diameter is the leaſt: And of the others, that which is nearer to that in which the Center is, is longer than one more diſtant.

3.

135. If Lines drawn from any Point without a Circle to the Circumference; then, of thoſe which fall on the concave Part of the Circumference, the greateſt will be that which paſſes through the Center, and the nearer to it greater than the more remote: But of thoſe which fall on the convex Part of the Circumference, the leaſt is that which coincides with the Diameter produced, and the nearer to it is leſs than the more remote.

4.

136. If a Point be taken within a Circle from which there fall more than two equal Lines to the Circumference, that Point is the Center of the Circle.

5.

137. One Circle cannot cut another in more than two Points.

6.

139. Equal Lines in a Circle are equally diſtant from the Center: And thoſe equally diſtant from the Center are equal to each other.

7.

*139. The Diameter is the greateſt Line in a Circle; and of any other, that which is nearer to the Center is always greater than that more remote.

8.

140. A Line greater than the Diameter drawn from any Point within the Circle muſt cut the Circumference.

9.

141. A Line meeting a Circle in a Point at Rt. ∠s to the Radius, is a Tangent to the Circle at that Point.

10.

142. If a Line touch a Circle in any Point, a Rt. Line drawn from the Center to that Point will be perpendicular to the touching Line.

11.

143. If a Line touches a Circle, and from the Point of Contact a Line be drawn perpendicular to the touching Line, the Center of the Circle ſhall be in that Line.

12.

145. The ∠ at the Center of a Circle is double of the ∠ at the Circumference upon the ſame Baſe; that is, upon the ſame Part of the Circumference.

13.

146. The ∠s in the ſame Segment of a Circle are = one another.

14.

147. The oppoſite ∠s of any quadrilateral Figure deſcribed in a Circle are together = 2 Rt. ∠s.

15.

148. The ∠ in a Semicircle is a Rt. ∠; but the ∠ in a Segment greater than a Semicircle is leſs than a Rt. ∠; and the ∠ in a Segment leſs than a Semicircle is greater than a Rt. ∠.

16.

149. In an oblique △ inſcribed in a Circle, the vertical ∠ exceeds, or is leſs than a Rt. ∠, by the ∠ made by the Baſe and the Diameter drawn from either Extremity of the Baſe.

17.

150. In equal Circles the Angles at the Circumference, and ſtanding upon equal Chords, are = each other: And if the Circumferences are equal, the Chords on which they ſtand will be equal alſo.

18.

151. If a Line touches a Circle, and from the Point of Contact a Line be drawn cutting the Circle, the ∠s made by this Line with the Line touching the Circle, ſhall be = the ∠s which are in the alternate Segment of the Circle.

19.

210. (See Pl. IV. Fig. 13.) 1ſt. As AB : BC :: AD : AC; 2d. As BD : DA :: DA : DC (this is all we mean when we ſay AD is a geometric mean Proportional between BD and DC.

20.

211. Therefore 1ſt. AB × AC = BC × AD; 2d. BD × DC = DA^2; 3d. BC × BD = AB^2; 4th. BC × DC = AC^2.

21.

212. If 2 Lines cut one another within a Circle, (as in Plate IV. Fig. 14.) the Rectangle of the Segments of one of them is = the Rectangle of the Segments of the other, *viz.* AE × EC = BE × ED. And if two Lines cutting a Circle, being produced, interſect each other without the Circle, (as in Pl. IV. Fig. 15.) then alſo will AE × EC = BE × ED.

22.

213. If from any Point C without a Circle, (Pl. IV. Fig. 16.) 2 Lines CA, CB, are drawn in ſuch a Manner that CB touches the Circle, and CA cuts it, then will $CA \times CD = BC^2$.

23.

214. In equal Circles, ∠s, whether at the Centers or Circumferences, have the ſame Ratio with the Circumferences on which they ſtand. Thus, let G and H (Pl. IV. F. 17.) be the Centers of the two Circles; then we affirm, that, as the Circumference BC : the Circumference EF :: the ∠ BGC : the ∠ EHF; and :: ∠ BAC : ∠ EDF.

24.

215. The ∠ BGC in the Center : 4 Rt. ∠s :: the Circumference or Arc BC on which it ſtands : the whole Circumference of the Circle.

25.

216. Hence the Circumferences DE, BC, (Pl. IV. Fig. 18.) of unequal Circles, which ſubtend equal Angles at the Centers, as ∠s DAE, BAC, are ſimilar; *viz.* As Arc DE : the whole Circumference of the Circle DEF :: BC : the Circumference of the Circle BCG. And if they are ſimilar, they will ſubtend equal Angles.

N. B. As to the Theorems relating to the Circumferences of Circles being as their Radii or Diameters. The Areas of ſimilar Plane Figures being as the Squares of their like Sides. &c. we intend to give them in a more proper Place, *viz.* when we treat on *Menſuration.*

XI. *Of Proportion.*

1.

187. Magnitudes have the ſame Ratio to one another which their Equimultiples have.

2.

188. If 4 Quantities are proportional, then any Equimultiples of the 2 firſt, will have the ſame Ratio as any Equimultiples of the two laſt.

3.

193. If 4 Lines are proportional, the Rectangle contained by the Extremes is = the Rectangle contained by the Means.

4.

198. If 4 Quantities are proportional, they will alſo be Proportionals taken inverſely, *viz.* if $a : b :: c : d$; then will $b : a :: d : c$.

5.

199, 200, 201, 202, 203. If 4 Quantities are proportional, they will alſo be proportional when taken alternately, compoundedly, by Diviſion, by Converſion, and mixtly, *viz.* if $a : b :: c : d$; then,

199. $a : c :: b : d$. Alternately.
200. $a + b : b :: c + d : d$. Compoundedly.
201. $a - b : b :: c - d : d$. By Diviſion.

202.

202. $a : a - b :: c : c - d$. By Converſion.

203. $a + b : a - b :: c + d : c - d$. Mixtly.

6.

205. If 3 Lines are proportional, the Rectangle under the Extremes is = the Square on the Mean, *viz.* if $a : b :: b : c$, then $ac = b^2$.

7.

206. The Rectangles under proportional Lines are proportional. Thus, if $a : b :: c : d$, and $e : f :: g : h$; then $a \times e : b \times f :: c \times g : d \times h$.

8.

207. The Squares on 4 proportional Lines are Proportionals, *viz.* if $a : b :: c : d$; then will $a^2 : b^2 :: c^2 : d^2$.

9.

208. If 4 Squares are Proportionals, their Sides will be ſo too.

A LIST *of the* PROBLEMS.

PROBLEM I.

111 *and* 112. To find the Areas of Parallelograms and Triangles.

2.

221. To draw a Line from a given Point = a given Line.

3.

222. From a greater Line to cut off a Part = a leſſer Line.

4.

223. About a given Point to deſcribe a Circle whoſe Radius ſhall be = a given Line.

5.

224. To make a △ whoſe Sides ſhall be reſpectively = three given Lines.

6.

225. To biſect a given ∠, that is, to divide it into two equal Parts.

7.

226. To biſect a given Line.

8.

227. The laſt Art. ſerves alſo to raiſe a Perpendicular on the Middle of a Line.

9.

228. To draw a Perpendicular to a given Line from any Point in it. See alſo Art. 315.

10.

229. To let fall a Perpendicular from a Point above to a Line below.

11.

230. To make an Angle = a given ∠.

12.

231. To conſtruct and ſhew the Uſe of a Protractor.

13.

13.

234. To conſtruct and ſhew the Uſe of a Line of Chords.

14.

236. To draw a Line parallel to a given Line through a given Point.

15.

237. At any Diſtance to draw a Line ∥ to another Line.

16.

239. To conſtruct a parallel Ruler.

17.

241. To triſect an Angle.

18.

242. To deſcribe a Parallelogram that ſhall be = a given △, and have one of its ∠s = a given ∠.

19.

243. To apply a Parallelogram to a given Line, which ſhall be = a given △, and have one of its ∠s = a given ∠.

20.

244. To deſcribe a Parallelogram = a given rectilineal Figure, and having an ∠ = a given rectilineal ∠.

21.

246. By this alſo may be deſcribed a Parallelogram = the Sum, or Difference, of any two given right-lined Figures.

22.

247. To deſcribe a Rectangle whoſe Sides ſhall be = two given Lines.

23.

248. Hence alſo to conſtruct a Square.

24.

249. To make a Square = any Number of given Squares.

25.

250. To make a Square = the Difference of two given Squares.

26.

251. To make a △ = any given quadrilateral Figure.

27.

252. To make a △ = any given 5-ſided Figure.

28.

253. To deſcribe a Circle through 3 given Points.

29.

254. To find the Center of a Circle.—— By having a Segment of a Circle to compleat the Circle.—— And to deſcribe a Circle about a given △.

30.

255. To draw a Tangent to a Circle from a given Point without it.

31.

256. To draw a Tangent from a given Point in the Circumference of the Circle.

32.

257. Upon a given Line to deſcribe a Segment of a Circle which ſhall contain a given Angle.

33.

259. To cut off a Segment from a given Circle which ſhall contain a given ∠.

34.

260. To inſcribe in a given Circle a Line = any given Line, leſs than the Diameter of the Circle.

35.

261. In a given Circle to inſcribe a △ equiangular to a given △.

36.

262. About a given Circle to deſcribe a △ equiangular to a given △.

37.

263. To inſcribe a Circle in a given △.

38.

264. To inſcribe a Square in a given Circle.

39.

265. To deſcribe a Square about a given Circle.

40.

266. To inſcribe a Circle in a given Square.

41.

267. To deſcribe a Circle about a given Square.

42.

275. In a given Circle to inſcribe a Polygon of a given Number of Sides.

43.

277. From a given Line to cut off any aliquot Part.

44.

278. To divide a given Line into any Number of equal Parts. Being of Uſe in making Scales of equal Parts.

45.

279. To find a third Proportional to two given Lines.

46.

280. To find a fourth Proportional to three given Lines.

47.

281. To find a mean Proportional to two given Lines.

48.

282. To divide a Line according to extreme and mean Proportion.

49.

283. To make a Square = any given right-line Figure.

50.

285. To find two Lines which ſhall have the ſame Ratio to each other as any two given rectilineal Figures.

51.

286. Upon a given Line to deſcribe a rectilineal Figure, ſimilar, and ſimilarly ſituated, to a given rectilineal Figure.

52.

288 To find the Height of an acceſſible Object by Means of two Poles of unequal Length.

53.

290. To find the Height of an acceſſible Object by Means of any plane Reflecting Surface, as a Looking-Glaſs, or Bowl of Water.

54.

291. To deſcribe a Figure called an Oval.

55.

292. To deſcribe a Figure called by ſome an Helix, or Spiral Line.

56.

293. Having a round Table, to make it into a ſquare Table, and have a Hole in the Middle, without Waſte.

57.

294. Having a round Table to change it into two oval-like Tables, and have a Hole in the Middle of each, without Waſte of Timber.

58.

295. To fill a Hole exactly a Foot ſquare, with a Piece of Board in Length 16 Inches, and Breadth 9 Inches, by only cutting it into two Parts.

59.

296. To deſcribe Circles of different Radii with the ſame Extent of the Compaſſes.

60.

297. To deſcribe an Oval-like Figure at one Sweep of the Compaſſes.

61.

298. To cut a Hole in a Card large enough for a Man to paſs through.

62.

299. To deſcribe an Ellipſis of any given Length and Breadth.

63.

300. To deſcribe a Figure reſembling a Hen's Egg.

64.

305. Any two Sides of a right-angled △ being given, to find the other Side.

As many valuable Authors in their Works frequently refer to ſome Propoſitions of *Euclid*, we have added the following Tables, to ſhew by Inſpection what Articles in this Eſſay anſwer to the moſt uſeful Theorems in that Author, and the contrary.

A TABLE ſhewing by Inſpection what Article in this Essay anſwers to its correſponding Propoſition in *Euclid*.

Propoſitions in *Euclid*.	Articles in this Eſſay.	Propoſitions in *Euclid*.	Articles in this Eſſay.	Propoſitions in *Euclid*.	Articles in this Eſſay.	Propoſitions in *Euclid*.	Articles in this Eſſay.
Book I. Prop.		Book II Prop.		Bo. III. Prop.		Bo. VI. Prop.	
2	221	1	314	1	260	1	189
3	222	2	313	2	261	2	191
4	58	4	119	3	262	3	192
6	62	5	121	4	263	4	194
8	95	11	282	5	254	5	197
9	225	14	283	6	264	6	196
10	226			7	265	8	209
11	228	Bo. III. Prop.		8	266	9	277
12	229			9	267	10	278
13	64	2	254	10	relate to deſcribing Polygons ſee Article 275.	11	279
15	69	3	134	11		12	280
18	89	7	*134	12		13	281
19	90	8	135	13		16	193
20	70	9	136	14		17	205
21	91	10	137	15		18	286
22	224	14	*138	16		33	214
23	230	15	139				
24	92	16	141				
25	93	17	255 256	Book V Prop.			
26	94						
27	73	18	142	4	188		
29	72	19	143	15	187		
30	74	20	145	16	199		
31	236	21	146	17	201		
32	76	22	147	18	200		
32	78	25	254				
1C.32	87	30	225				
2C.32	88	31	148				
34	97	32	151				
35	102	33	257				
36	105	34	259				
41	106	35	212				
42	242	36	213				
43	113						
44	243						
45	244						
46	248						
47	115						

A TABLE ſhewing what Propoſition in *Euclid* anſwers to its correſponding Article in this Essay.

Art. in this Eſſay.	Propoſ. in *Euclid.*		Art. in this Eſſay.	Propoſ. in *Euclid.*		Art. in this Eſſay.	Propoſ. in *Eulid.*		Art. in this Eſſay.	Propoſ. in *Euclid.*	
	Prop.	Book		Prop.	Book		Prop.	Book		Prop.	Book
58	4.	1	119	4.	2	199	16.	5	261	2.	4
59	5.	1	121	5.	2	200	18.	5	262	3.	4
62	6.	1	134	3.	3	201	17.	5	263	4.	4
64	13.	1	*134	7.	3	205	17.	6	264	6.	4
69	15.	1	135	8.	3	209	8.	6	265	7.	4
70	20.	1	136	9.	3	212	35.	3	266	8.	4
72	29.	1	137	10.	3	213	36.	3	267	9.	4
73	27.	1	*138	14.	3	214	33.	6	275 ſhews how to deſcribe Polygons, which *Euclid* does in 10, 11, 12, 13, 14, 15, 16, of 4th Book, but in a Method entirely different.		
74	30.	1	139	15.	3	221	2.	1			
76	32.	1	141	16.	3	222	3.	1			
78	32.	1	142	18.	3	224	22.	1			
87	1C.32.1.		143	19.	3	225	9.	1			
88	2C.32.1.		145	20.	3	226	10.	1			
89	18.	1	146	21.	3	228	11.	1			
90	19.	1	147	22.	3	229	12.	1			
91	21.	1	148	31.	3	230	23.	1			
92	24.	1	151	32.	3	236	31.	1			
93	25.	1	187	15.	5	242	42.	1	277	9.	6
94	26.	1	188	4.	5	243	44.	1	278	10.	6
95	8.	1	189	1.	6	244	45.	1	279	11.	6
97	34.	1	191	2.	6	248	46.	1	280	12.	6
102	35.	1	192	3.	6	254	2.	3	281	13.	6
105	36.	1	193	16.	6		5.	4	282	11.	6
106	41.	1	194	4.	6	255	17.	3	283	14.	2
113	43.	1	196	6.	6	256			286	18.	6
115	47.	1	197	5.	6	257	33.	3	313	2.	2
						259	34.	3	314	1.	2
						260	1.	4			

The END.

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AN

ESSAY

ON THE

ELEMENTS

OF

Plane Trigonometry.

With their APPLICATION to ALTIMETRY and LONGIMETRY.

BY BENJAMIN DONN.

Trigonometria plana eſt ſcientia ex tribus trianguli rectilinei partibus inveniendi reliquas.

The PREFACE.

HAVING already given the Elements of Geometry, we come now to treat of Plane Trigonometry.

It is by Trigonometry that we apply Geometry to find the Heights of Objects and Diſtances of Places, both on Sea and Land, as is ſhewn in this Eſſay. Hence it appears, that, even in the narrow Limits to which it is here confined, it is a Science of great Uſe in the common Affairs of Life; and, if we include the Doctrine of ſpherical Triangles alſo, under the Term Trigonometry, it is that Science to which may be attributed whatever is effected in Aſtronomy, Geography, and Navigation, agreeably to the Motto prefixed to this Volume. For by Trigonometry it is the Aſtronomer computes the Diſtances of the Planets, calculates the Eclipſes of the Sun and Moon, the Tranſits of Venus and Mercury, conſtructs Dials for ſhewing the true Time of the Day, *&c.* By it the Geographer lays down the principal Places on the terraqueous Globe in their true Poſitions and Diſtances: And, laſtly, by it the Navigator conducts his Ship, through the trackleſs Ocean, to his deſired Port, though ever ſo far diſtant, by the uncertain Impulſe of the Wind.

What we have attempted to do, in this Eſſay, is, to give the Elements of Plane Trigonometry in a clear Manner, with their Application to Altimetry and Longimetry, in a greater Variety of uſeful Problems, both for Land and Maritime Surveying, than is given in any other Treatiſe on this Subject.

And here it may not be improper to hint, that no Perſon, though accuſtomed to conſtruct Plans of Gentlemens Eſtates by the Theodolite, &c. ſhould attempt to ſurvey a County, or the Sea-Coaſts or Channels of any Kingdom, till he has made himſelf Maſter of theſe Problems, with as much of Navigation or Aſtronomy, at leaſt, as teaches to find the Latitudes and Longitudes of Places; which done, for his Encouragement, we will venture to affirm there will not ariſe any Difficulty, in Practice, which he may not eaſily ſurmount.

The Writers on this ſubject are numerous; as almoſt all Authors, who have treated of Surveying, Navigation, &c. have written preparatory Treatiſes on Plane Trigonometry: Some of them are, *Pitiſcus*, in the Year 1614; *Gunter*, 1623; *Briggs*, 1633; *Newton*, 1658; *Sturmy*, 1667; *Norwood*, 1631; *Seller*, 1669; *Dechales*, 1674; *Phillipps*, 1678; Sir *Jonas Moore*, 1681; *Atkinſon*, 1686; *Leybourn*, 1690; *M. Bouger*, 1694; *Leybourn*, 1704; *Sherwin*, 1706; *Harris*, 1706; *Hodgſon*, 1706; *Ozanam*, 1718; *Laurence*, 1717; *Wilſon*, 1720; *Hawney*, 1725; *Keil*, 1728; *Wolfius*, 1732; *Kelly*, 1734; *Martin*, 1736; *Hauxley*, 1743; *Simpſon*, 1748; *Muller*, 1748; *Emerſon*, 1749; *Barrow*, 1750; *Webſter*, 1751; *M. Bouger*, 1753; *Holliday*, 1756; *Martin*, 1759; *Maſeres*, 1760; *Mountaine*, 1763; *Robertſon*, 1764; *Patoun*, 1765; *Simſon*, 1767; *Hutton*, 1770; *Payne*, 1772.

A
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THE ELEMENTS OF Plane Trigonometry.

BOOK I. CHAP. I.

Of DEFINITIONS, *and the Nature of* SINES, TANGENTS, *&c.*

1. TRIGONOMETRY is the Science or Doctrine of the Properties of Triangles.

2. A *Triangle* is a Figure contained under three Lines.

3. Trigonometry may be either *Plane* or *Spherical.* Plane Trigonometry is, when the three Lines which form the Triangle are Right Lines. It is called *Plane*, becauſe theſe Lines are in the ſame Plane, or flat Surface. By ſome Authors it is called *Plain*, becauſe the common Principles of Plane are Something eaſier to be underſtood than thoſe of Spherical Trigonometry.

4. *Spherical Trigonometry* treats of the Properties of Triangles, formed, or conceived to be formed, on the Superficies of a Globe or Sphere; which are therefore called Spherical Triangles. But this muſt be the Subject of ſome future Eſſay.

Of the Chord, Sine, Tangent, &c. of an Arc.

Pl. I. Fig. 1, 2. 5 Let ADB be a Semicircle, C the Center, the Arcs AD and DB each *90 Degrees, or a Quarter of a Circle. * 217. g.

Firſt. Aſſume any Point, F, in the Arc BD; join B, F; then the Line BF is called the *Chord* of the Arc BF.

6. *Secondly.* From the Point F, let fall the Line FE, perpendicular to AB; or (which is the ſame) draw FE parallel to DC; then is the Line FE called the *Sine*, and EB the *Verſed-Sine*, of the Arc BF.

7. *Thirdly.* At the Point B, on AC, raiſe the Perpendicular BG; join the Points, C, F, by the Line CF; which produce, to interſect BG in G; then is BG called the *Tangent*, and CG the *Secant*, of the Arc BF.

8. *Fourthly.* Any Arc, BF, leſs than a Quarter of a Circle, being ſubtracted from 90 Degrees, or the Quarter of the Circle DB, the remaining Arc FD is called the *Complement* of the Arc BF.

9. *Fifthly.* From the Point, F, draw FI, perpendicular to CD; which is eaſieſt done by drawing it parallel to AB: Alſo, from the Point D, DH parallel to AB, (or, which is the ſame, perpendicular to DC,) till it interſects CG in H: Then is IF the *Sine-Complement*, DH the *Tangent-Complement*, and CH the *Secant-Complement*, of the Arc FB; or (which is the ſame Thing) the *Sine*, *Tangent*, and *Secant*, of the Arc DF.

10. Inſtead of Sine-Complement, Tangent-Complement, and Secant-Complement, it is very common to write Co-ſine, Co-tangent, and Co-ſecant.

11. By the Conſtruction it appears that CE is always equal to IF, the *Coſine* of the Arc BF.

---

* The Reference, 217. g. at this Mark in the Margin, ſignifies, Article 217, Elements of Geometry: And, in general, when g. is added to a Reference, it denotes the Elements of Geometry; otherwiſe, the Article in this Eſſay.

12. Hence it appears, that any Arc, BF, and its *Supplement* to 180 Degrees, (*viz.* the Arc ADF,) have the ſame Sine FE, Tangent BG, and Secant CG, but not the ſame Verſed-Sine; BE being the Verſed-Sine of the Arc BF, and AE that of the Arc ADF.

13. It appears, from Figure II. that, when the Arc BF is equal to 45 Degrees, the Radius CB is equal to the Tangent BG.

For, as the Triangle CBG is right-angled at B, and the ∠BCG being 45 Degrees by the Hypotheſis, the Angle CGB is alſo [a] 45 Degrees; and therefore the Sides, CB, BG, being oppoſite to equal Angles, muſt be [b] equal.

[a] 83. g.

[b] 62. g.

14. As Arcs are the Meaſures of Angles, it is as cuſtomary to ſay, the Sine, Tangent, or Secant, of an Angle, as the Sine, Tangent, *&c.* of an Arc.

15. Before we proceed any farther, it may be proper to inſert the Explanation of ſuch Characters as will be uſed in this Eſſay, which are not already explained in our *Elements of Geometry.* Viz.

CHARACTERS.

S.	Sine.
S. C.	Sine-Complement.
C. S.	Coſine.
T.	Tangent.
T. C.	Tangent-Complement.
Sec.	Secant.
V. S.	Verſed-Sine.
Sup.	Supplement to 180 Degrees.
d. *or* °	Degrees } as 3° 17′ 14″ is to be read 3 Degrees, 17 Minutes, 14 Seconds.
′	Minutes
″	Seconds
h.	Hours, as 5h. 10′ is 5 Hours, 10 Minutes.

16. Before we proceed to the Solutions of the common Caſes of Plane Triangles, it is proper to explain the Conſtruction of the ſeveral Lines gene-

rally put on a Plotting-Scale, *viz.* Lines or Scales of equal Parts, alſo of Chords, Sines, Tangents, and Secants, for the more readily conſtructing Triangles, *&c.*

BOOK I. CHAP. II.

Of the Conſtruction of Scales.

17. A SCALE of 10 equal Parts in an Inch, as AD, may be conſtructed by a few Trials (or by Art. 278, of *Geometry*).

18. If it be required to make any other Scale of equal Parts, (as, for Example, 15 to an Inch,) it may be done thus.

Pl. I. Fig. 3. Firſt, Draw AC ⊥ to AB, and from any convenient Place, C, draw CF, making any convenient Angle, at Pleaſure, with the Line CA; join C, D, and make CI = AD; then take from the Scale, AB, the Number of Parts into which the new Scale is to be divided; *viz.* in this Example, extend the Compaſſes from E to *a*, and ſet that Extent from C on the Line CA (produced if neceſſary) to K, and join IK, and draw AL || to KI; make CM = CL, and draw MN || AB; then will M*b* be = one of the primary Diviſions, on the New Scale.*

---

* DEMONSTRATION. Since the Number of Diviſions is to be greater in an Inch Length, in the Scale MN, than in the Scale AB, it is manifeſt that the primary Diviſions muſt be decreaſed in that Proportion; that is, AD muſt be, in this Example, to M*b*, as 15 to 10.

That it is ſo, in the above Conſtruction, will eaſily appear; for CK was taken to CI in that Proportion; and, AL being || to IK, the △s CIK and CLA are ſimilar; ∵ CL has the ſame Ratio to CA as [a] CI to CK. But CM is made = CL; ∵ CM : CA :: CI : CK. But, M*b* being || AD, the △s CM*b* and CAD are ſimilar. ∵ CM : CA :: M*b* : AD; ∵ by Equality, M*b* : AD as CI to CK, the given Proportion. Q. E. D.

[a] 191. g.

After this, that M*b*, by drawing Lines from C to the ſeveral Diviſions in AD, will be divided into 10 equal Parts; is too manifeſt to need any Explanation.

Then Lines drawn from the Point C, to the ſeveral Diviſions in AD, will divide M*b* into 10 equal Parts.

19. *The Uſe of a Scale of equal Parts, as repreſented by* AB.

The primary Diviſion 1 may be conſidered either as $\frac{1}{10}$, 1, or 10, or 100, *&c.* and then the Diviſion 2 muſt be reckoned proportional; as, if 1 is $\frac{1}{10}$, 2 is $\frac{2}{10}$; if 1 be 1, 2 is 2; if 1 ſtands for 10, 2 is for 20; if 1 for 100, 2 for 200, *&c.* And as the Diviſions on the Left, *viz.* AD, are divided into 10 equal Parts, each is $\frac{1}{10}$ of what the firſt primary Diviſion, DE, is ſuppoſed to be.

For Example. Let it be required to take off $\frac{5}{10}$ from the Scale, ſuppoſing each primary Diviſion to be 1. Place one Point of the Compaſſes on D; then open the other to the fifth Diviſion at *a*; and the Compaſſes will be opened to $\frac{5}{10}$.

Again: Let it be required to take off 25, ſuppoſing the Diviſion at E to denote 10. Extend the Compaſſes from B to *a*, and it will be the Diſtance required.

The ſame Extent, if 1 at E be conſidered as 100, will denote 250.

20. But, when it is required to take off a Number expreſſed by three or more Figures pretty accurately, it is more convenient to have a *Diagonal Scale*; which is thus conſtructed.

For Example's Sake, let it be required to conſtruct a Scale for dividing an Inch into 100 equal Parts.

Pl. I. Fig 4. Having drawn *ab*, *cd*, and made the primary Diviſions, conſider how many Parts you would have a primary Diviſion, *ae*, divided into; ſuppoſe, as in the common Scales, 100; then the Number of Diviſions in *ce*, × by the Number of Diviſions in *cm*, muſt be equal to 100; and ∵ each is generally made to contain 10 Diviſions, 10 Times 10 being 100. Through the ſeveral Diviſions in *cm*, and parallel to *cd*, draw the 10 parallel Lines, as repreſented in the Figure: Divide *ce* and *mn* each into 10 equal Parts, and join the Points by the Diagonal Lines, as ſhewn in the Figure.

21. An Example or two will explain the Uſe of this Scale.

Firſt, ſuppoſe it is required to take off 156 in the Compaſſes, the primary Diviſion being 100.

Put one Point of the Compaſſes at the Place where the 6th || Line interſects the vertical Line of the firſt primary Diviſion, *viz.* at *o*, and extend the other Leg to the Place where the 5th Diagonal Line cuts the ſame Parallel; and the Thing is done.

22. Example 2. Suppoſe it is required to take off 2740, each primary Diviſion being 1000.

Place one Leg of the Compaſſes at the Place where the 4th Parallel interſects the 2d primary Diviſion, *viz.* at *r*, and extend the other on the ſame Parallel, to the 7th Diagonal; and the Extent of the Compaſſes will be the Diſtance required.

Hence the Learner may obſerve, that the parallel Diviſions in the Diagonal Scale are conſidered but as $\frac{1}{10}$ of the diagonal Diviſions.

23. *Scholium.* For other Purpoſes, the Diagonal Scale muſt be otherwiſe divided.

For Example, if to plot Diſtances taken by a Perambulator, or Meaſuring-Wheel, which takes Miles, Half-Miles, and Poles, (which I have found to be beſt for ſurveying Counties,) then a Mile containing 320 Poles, I conſtruct it thus: Make $ae = \frac{1}{2}$ a primary Diviſion, or Half a Mile; then divide it into 16 equal Parts, and *cm* into 10; for $16 \times 10 = 160$ $= \frac{1}{2}$ a Mile.

But if the Wheel takes Miles, Furlongs, and Poles, as ſome are made, (which I would not recommend, for Reaſons which will appear in a more proper Place,) then I divide each primary Diviſion into 8 Parts, for Furlongs, and draw vertical Lines, as in the primary Diviſions; after which I make *ce* = the Diſtance for a Furlong, and divide it into 4 equal Parts, and *em* into 10; becauſe 4 × 10 = 40 Poles = 1 Furlong.

In my Survey of *Devon*, I had both theſe Kinds of Wheels; which occaſioned my conſtructing Scales on theſe two Methods; and, as young Surveyors might perhaps be at ſome Loſs to contrive them, I take this Opportunity to hint at them here, rather than wait till our Treatiſe of Surveying, as they are not conſtructed in any Book on that Subject.

24. *To conſtruct Scales of Chords, Sines, Tangents*, &c.

1ſt. With a convenient Extent of the Compaſſes, and one Point at C, deſcribe a Semicircle, ADB, Pl. I. Fig. 5. and raiſe the $\perp$ CD; then with the ſame Extent of the Compaſſes, and one Point in D, with the other make a Mark, H, on the Arc DGHB; then, with the ſame Extent, one Leg being in B, the other will extend to G, and the Arc DB be divided into 3 equal Parts; and $\because$ DG, GH, and HB, are each = [a] 30 Degrees; then, by a Trial or two, divide [a] 276. g. each of theſe into three equal Parts, and ſo the Quadrant will be divided into 9 equal Parts, each 10 Degrees.

2. Produce AB towards F at Pleaſure, and draw DE $\parallel$ thereto; then, turning a Ruler about C as a Center, draw Lines to touch the Tangent Line DE, *viz.* *a* 10, *b* 20, G 30, *&c.* then it is manifeſt, from Art. 7, that DE will be a Scale of Tangents.

3. Then, one Leg of the Compaſſes being in C, open the other to 10 on the Tangent-Line, and, turning it about, ſweep an Arc, 10, 10, to cut the Line CF; then ſweep Arcs, 20, 20; 30, 30; *&c.* and CF becomes a Scale of Secants.

4. Parallel to DC draw the Parallels, *a* 10, *b* 20, *&c.* to cut the Line CB; then is CB a Scale of Sines; and, if the Numbers were counted from B towards C, a Scale of *V*erſed Sines, by Art. 6.

5. Place one Leg of the Compaſſes in D, and, extending the other to *a*, deſcribe the Arc *a* 10; in the ſame Manner, *b* 20, *&c.* to cut the Line DB; then will the Line DB be a Line of Chords, by Art. 5.

6. Laſtly. If A, D, be joined by a Line, AD, and the Arc AD divided into 8 equal Parts, denoting Points of the Compaſs uſed in Navigation, and Arcs 1, 1; 2, 2; *&c.* be deſcribed in the ſame Manner as in the Line of Chords; the Line AD becomes a Scale of Chords for Points, commonly called, by Sailors, a Line of Rumbs.

25. Though, in the above, we have only divided the Arc DB into 9 Parts, denoting 10 Degrees each, (being ſufficient for an Example,) it is eaſy to conceive that each of theſe may be divided into 10 Degrees, and ſo the Arc divided into 90 Degrees, and conſequently the Scales made to Degrees, *&c.*

26. But, though the above Conſtruction clearly ſhews the Nature of ſuch Scales, yet, as it requires very great Care to be accurate, it is not ſo fit for Practice as the conſtructing them by Means of Tables, *&c.* as I ſhall ſhew farther on, being unwilling to detain the young Student any longer in this Place, and deſirous of making him acquainted with the Solution of the common Caſes of Plane Triangles, as ſoon as conveniently may be.

27. Having a Set of Scales, accurately made, called Pattern-Scales, readily to make others by it.

For Example: Let it be required to transfer the Graduations on the Scale AB, to the intended Scale DC. Pl. II. Fig. 1.

1ſt. Draw the neceſſary parallel Lines from D to C, by the ſharp Point of a Gage, of the ſame Nature as thoſe uſed by the Cabinet-makers or Joiners.

2. Fix the two Scales by the Side of each other, on a Table or proper Board, ſo that they may not ſlide.

3. Then take a Square, *abe*, in which *ab* is a Ruler, or Piece of Mahogany or other proper Wood, in the Middle of which is laid a Braſs Square, *cde*.

Its Uſe is: Slide *ab* along the Side of the Pattern-Scale AB, till the Edge *de* comes to a Diviſion, ſuppoſe 2, to be transferred to the other Scale; which is readily done by only cutting with a ſharp-pointed Knife, by the Edge *de* of the Square on the Scale DC; and ſo ſliding from one Graduation to another, the Diviſions on AB are eaſily copied on the intended Scale DC. As to the Figures, if on Wood, they are marked by proper Stamps; if on Braſs, they ſhould be engraved.

28. To make an ∠ = any Number of Degrees.

This has been already done by a Line of Chords, and may alſo be done by Scales of Sines, Tangents, or Secants.

For Example: Let it be required to make an ∠ equal to 50 Degrees.

1ſt. By the Line of Sines.

Pl. II. F. 2. On any Point, C, in the Line AB, raiſe the ⊥ CD = the Sine of 50d. taken from the Scale of Sines; then, from the ſame Scale, take the Radius or Sine of 90d. and, having one Point of the Compaſſes in D as a Center, with the other interſect AB in E, by a little Sweep; join ED; and the ∠DEB is manifeſtly that required.

2. By the Line of Tangents.

Pl. II. F. 3. Take AC = the Radius or Tangent of 45d. from the Scale of Tangents; raiſe the ⊥ CD, which make = the Tangent of 50 Degrees; draw AD; and the ∠ DAB is that required.

3. By the Secants.

Pl. II. F. 4. Take, as before, AC = Radius, (= Secant of 0 Degrees = Sine of 60 d. = Tangent of 45°,) and raiſe ⊥ CE; then with the Secant of 50 d. and one Leg of the Compaſſes in A, with the other ſweep a little Arc, interſecting CE in D; and it is evident the ∠ DAB is = the required ∠.

BOOK I. CHAP. III.

Of the Solution of the common Caſes of Right-angled Plane Triangles.

29. *Definition.* In any right-angled Triangle, ABC, right-angled at B, the Side AB, on which the Triangle may be ſuppoſed to ſtand, is called the Baſe; and the Side BC, from its Poſition, the Perpendicular. The Side oppoſite the Right-Angle, or the longeſt Side, AC, is named the Hypothenuſe. Pl. II. Fig. 5.

30. By attentively conſidering Article 4, theſe Propoſitions evidently follow; *viz.*

1ſt. If the Hypothenuſe be made Radius, then is the Perpendicular BC the Sine of the oppoſite Angle A; and the Baſe AB the Coſine of the Angle A; or, which is the ſame, the Sine of the Angle C.

Pl. II. Fig. 6. 2dly. If the Baſe be made Radius, then is the Perpendicular the Tangent of the Angle A, and the Hypothenuſe the Secant of the Angle A.

Pl. II. Fig. 7. 3dly. If the Perpendicular be made Radius, then is the Baſe the Tangent of the Angle C, and the Hypothenuſe the Secant of the Angle C.

31. *Scholium.* We have noted the above, as we intend to ſhew how to ſolve right-angled Triangles by making each Side Radius, as ſome may expect it; but we ſhould hint, that, in Practice, Navigators, Surveyors, *&c.* generally make the Hypothenuſe Radius, when it can be done (which it may always, excepting when the two ſhort Sides or Legs, *viz.* the Baſe and the Perpendicular, are given to find the Hypothenuſe); becauſe it is more eaſily remembered, *&c.* as it comes under the following univerſal Theorem,* *viz.* In any Plane Triangle, if

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* Which will be demonſtrated in Oblique Triangles.

among the Things given there be a Side, and its oppoſite Angle, to find the Reſt: then ſay, As a given Side : the Sine of its oppoſite Angle :: any other Side : the Sine of its oppoſite Angle.

Therefore, to find an Angle, begin with a Side oppoſite to a known Angle. Alſo, as the Sine of a given Angle : its oppoſite Side :: the Sine of any other Angle : its oppoſite Side.

Therefore, to find a Side, begin with an Angle oppoſite to a known Side.

32. *Corollary* 1. Hence, if the oppoſite Sides are equal, the oppoſite Angles are equal; and the contrary.

33. *Corollary* 2. Hence alſo the greater Angle is oppoſite to the greater Side, and the contrary; agreeably to Theorem the 6th, in the Introduction.

34. The ſix Caſes of Right-angled Trigonometry may be eaſily ſolved, by obſerving the following Directions: *Viz.*

1. Either make a Draught of the Triangle by Geometry, or (which may be ſufficient) only a Repreſentation, drawn in a rough Manner by Hand.

2. Let ſuch Parts of the Triangle as are given in the Queſtion be marked with a Daſh (—), and thoſe which are to be found, with a Cypher (o).

3. If one of the Acute-Angles is given, the other is alſo looked on as known; being found by only ſubtracting the given one from 90 d. This appears from Art. 83, *Elements of Geometry*.

4. Compare the given Things together, and thereby determine which of the preceding Rules is beſt adapted to ſolve the Problem.

5. Then write the Canon as the Theorem directs; and, if you uſe a Table of Natural Sines, Tangents, and Secants, work the Proportion as in the Rule of Three in common Arithmetic; but, if a Table of the Logarithms of theſe Natural Sines, Tangents, &c. place the Canon in four Lines under one another; and, taking out of the proper Ta-

bles the Logarithms of the given Quantities, add the Logarithms of the ſecond and third Terms together, and ſubtract the Logarithm of the firſt from the Sum, and the Remainder will be the Logarithm of the fourth Term.

For Logarithms are Numbers artificially contrived, in ſuch a Manner, that their Addition anſwers to Multiplication, and Subtraction of them to Diviſion of Natural Numbers.

35. *N. B.* The Index of the Logarithm of any Number is always one leſs than the Number of Integral-Places in the given Number; and, conſequently, the Number of Integral-Places of Figures in any Number is always one more than the Index of its Logarithm.

If the young Student is inclined to ſee the Doctrine of Logarithms more fully treated of, he may conſult our Eſſay on that Subject. The Tables of Logarithms are too voluminous to be given in this Volume, as they cannot conveniently be comprized in leſs than 135 Pages; therefore, if the young Student has not a good Set already by him, we muſt refer him to our *Britiſh Mariner's Aſſiſtant*, juſt publiſhed, (Price Six Shillings,) where, amongſt 40 Tables adapted to the ſeveral Purpoſes of Trigonometry and Navigation, he will find very accurate Tables of Logarithms to *ſeven* Places of Decimals. However, as we may perhaps frequently have Occaſion to ſolve ſome practical Problem in Trigonometry, when Tables of Logarithms are not at Hand, I have added, at the End of this Eſſay, a Table of Natural Sines, Tangents, and Secants, to 4 Figures; and, if we have Occaſion to ſolve a common Queſtion in the Abſence of any Books, (which may ſometimes be the Caſe,) I have contrived a Nautical Pocket-Piece, about the Size of a Crown, ſhewing the Natural Sines and Tangents to 3 Places, which is ſufficiently exact for moſt Uſes at Sea, *&c.*

to be had of the Publiſher of this Eſſay, Price Sixpence.

36. The common Caſes of right-angled △s are:

Caſe 1. Given the Angle at the Baſe, and the Hypothenuſe, to find the Baſe and the Perpendicular.

Example. Let the Angle at A be 40 d. 15′, and the Hypothenuſe 130 Feet: Quære the other two Sides? Pl. II. Fig. 8.

Conſtruction 1. Draw a Line, AB, at Pleaſure.

2. Make an ∠ A = the given ∠ 40 d. 15′, and draw AC, which make equal to the Hypothenuſe, 130 Feet.

3. From C let fall the Perpendicular CB; then it is manifeſt that the right-angled Triangle ABC is that required.

37. If AB and BC be meaſured on the ſame Scale of equal Parts as AC was ſet off from, you will have nearly the Lengths of the Baſe and Perpendicular; but more accurately by Calculation, as follows.

CANONS.

38. 1ſt. Making the Hypothenuſe Radius.

To find the Baſe.		To find the Perpendicular.	
As Rad. or S. of 90°	10.0000000	As Radius 90°	10.0000000
Is to AC 130 Feet	2.1139433	Is to AC 130 Feet	2.1139433
So is S. ∠ C 49° 45′	9.8826568	So is S. ∠ A 40° 15′	9.8103159
To AB 99.22 Feet	~~1~~1.9966001	To BC 84 Feet	~~1~~1.9242592

By Natural Sines thus. By the Table, at the End of this Eſſay, the Sine of 49° 45′ is .7632, and the Sine of 40° 15′ is .6461; therefore ſay, by the Golden Rule, As Radius 1 : 130 :: .7632 : .7632 × .130 = 99.2, nearly, the required Baſe; and as 1 : 130 :: .6461 × 130 = 84, nearly, the Perpendicular which was to be found.

39. Note, ∠ C 49 d. 45′ was found by ſubtracting ∠ A 40 d. 15′ from 90 d. If the Learner is inclined,

clined, he may ſave himſelf the Trouble of ſubtracting, by taking out the Coſine of 40 d. 15′; becauſe, the Angles A and C being always = 90 d. the Coſine of the ∠ A muſt be equal to the Sine of the Angle C.

40. In the above Canons, the firſt Logarithm is to be ſubtracted from the Sum of the two middle Logarithms; which, as the firſt Logarithm is 10, is here done by only cutting off 1, in the Place of Tens, in the Sum.

41. 2. Making the Baſe Radius.

As Sec. ∠A* 40° 15′	10.1173432	As S. ∠A 40° 15′	10.1173432
Is to AC 130 Feet	2.1139433	Is to AC 130 Feet	2.1139433
So is Radius 90°	10.0000000	So Tan. ∠A 40° 15′	9.9276590
	12.1139433		12.0416023
	10.1173432		10.1173432
To AB 99.22 Feet	1.9966001	To BC 84 Feet	1.9242591

Or thus, by the Table of Natural Secants, &c.

By the Table, the Natural Secant of 40° 15′ = 1.309
And the Natural Tangent of 40 15 = .8466

Then, as 1.309 : 130 :: 1 : $\frac{130 \times 1}{1.309}$ = 99.3, nearly, the Baſe required.

Again, as 1.309 : 130 :: 0.8466 : $\frac{0.8466 \times 130}{1.309}$ = 84, nearly, = the required Perpendicular.

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* If the Table of Logarithms we are uſing has not a Table of Secants, take out the Log. Coſine of 40° 15′, *viz.* 9.8826568, and, ſubtracting it from 20, we readily get 10.1173432, the required Log. Secant of 40° 15′. And in the ſame Manner the Secant for any other Number of Degrees may be found.

42. 3. Making the Perpendicular Radius.

As Sec. ∠C 49°45′	10.1896841	As Sec. ∠C 49°45′	10.1896841
Is to AC 130 Feet	2.1139433	Is to AC 130 Feet	2.1139433
So Tan. ∠C 49°45′	10.0723410	So is Radius 90°	10.0000000
	12.1862843		12.1139433
	10.1896841		10.1896841
To AB 99.22 Feet	1.9966002	To BC 84 Feet	1.9242592

Or thus.

The Natural Secant of 49° 45′ being 1.547
And the Natural Tangent of 49 45 being 1.181

We have, As 1.547 : 130 :: 1.181 : $\frac{1.181 \times 130}{1.547}$ = 99.2, nearly, = the required Baſe.

And, laſtly, As 1.547 : 130 :: 1 : $\frac{130 \times 1}{1.547}$ = 84, nearly, = the Perpendicular which was required.

43. In like Manner may the following Caſes be ſolved by Natural Sines, Tangents, &c. as well as by Logarithms; but as, in Order to be very nearly accurate, Tables muſt be given, calculated to every Minute of the Quadrant or 90 Degrees, it would take up more Room than we could ſpare; and, as they muſt be computed to 6 or 7 Places of Decimals, the Operations of multiplying and dividing would become more tedious than by Logarithms: We ſhall, therefore, for the Future, give only the Solution by Logarithms, it being ſufficient to have ſhewn the Method of Solution by Natural Numbers.

N. B. The Natural Sines are given to 7 Places of Figures, for every Minute of the Quadrant, in our *Britiſh Mariner's Aſſiſtant.*

44. *Caſe* 2. Given the Angle at the Baſe, and the Baſe, to find the Perpendicular and Hypothenuſe.

Example. Let the Angle at A be 50 d. and the Baſe 86 Yards; to find the Perpendicular and Hypothenuſe.

Pl. II. Fig. 9.

Conſtruction. Make ∠ A = 50 d. AB = Baſe, 86 Yards: On the Point B erect the Perpendicular BC, to interſect AC in C: Then is ABC the Triangle required.

Calculation. — 1. Making the Hypothenuſe AC Radius.

45. To find the Perpendicular.		To find the Hypothenuſe.	
As S. ∠ C 40°	9.8080675	As S. ∠C 40°	9.8080675
Is to AB 86 Yards	1.9344984	Is to AB 86 Yards	1.9344984
So is S. ∠ A 50°	9.8842540	So is Radius 90°	10.0000000
	11.8187524		11.9344984
	9.8080675		9.8080675
To BC 102.5 Yards	2.0106849	To AC 133.8 Yards	2.1264309

46. 2. Making the Perpendicular Radius.

As Tang. ∠ C 40°	9.9238135	As Tang. ∠ C 40°	9.9238135
Is to AB 86 Yards	1.9344984	Is to AB 86 Yards	1.9344984
So is Radius 90°	10.0000000	So is Sec. ∠C 40°	10.1157460
	11.9344984		12.0502444
	9.9238135		9.9238135
To BC 102.5 Yards	2.0106849	To AC 133.8 Yards	2.1264309

47. 3. Making the Baſe Radius.

As Radius 90°	10.0000000	As Radius 90°	10.0000000
Is to AB 86 Yards	1.9344984	Is to AB 86 Yards	1.9344984
So is Tan. ∠ A 50°	10.0761865	So is Sec. ∠ A 50°	10.1919325
To BC 102.5 Yds	~~1~~2.0106849	To AC 133.8 Yds	~~1~~2.1264309

48. *Caſe* 3. Given the Perpendicular and the Angle at the Baſe, to find the Baſe and Hypothenuſe.

Example. Let the ∠ at A be 33 d. 45′, and the ⊥ 75 Feet; required the Baſe and Hypothenuſe. Pl. II. Fig. 10.

Conſtruction. Make ∠ A = 33 d. 45′, and draw AD; then, ∥ to AB, at a Diſtance = 75 Feet, draw EF, to interſect AD in C; from C let fall the ⊥ CB: Then it is evident the Δ ABC is that required.

Otherwiſe, 1ſt, By ſubtracting the ∠ A 33 d. 45′ from 90 d. find the ∠ C = 56 d. 15′. 2. Aſſume a Point, B, in the Line AG, and raiſe the ⊥ BC = 75 Feet. 3. Make ∠ G = 56 d. 15′, and draw CA, to interſect HG in A; and the Δ is conſtructed.

Calculation. —— 1. Making the Hypothenuſe Radius.

49. To find the Baſe.		To find the Hypothenuſe.	
As S. ∠ A 33° 45′	9.7447390	As S. ∠ A 33° 45′	9.7447390
Is to BC 75 Feet	1.8750613	Is to BC 75 Feet	1.8750613
So is S. ∠ C 56° 15′	9.9198464	So is Radius 90°	10.0000000
	11.7949077		11.8750613
	9.7447390		9.7447390
To AB 112.3 Feet	2.0501687	To AC 135 Feet	2.1303223

50. 2. Making the Perpendicular Radius.

As Radius 90°	10.0000000	As Radius 90°	10.0000000
Is to BC 75 Feet	1.8750613	Is to BC 75 Feet	1.8750613
So Tan. ∠C 56°15'	10.1751074	So Sec. ∠C 56°15'	10.2552610
To AB 112.3 Feet	12.0501687	To AC 135 Feet	~~1~~2.1303223

51. 3. Making the Baſe Radius.

As Tan. ∠A 33°45'	9.8248926	As Tan. ∠A 33°45'	9.8248926
Is to BC 75 Feet	1.8750613	Is to BC 75 Feet	1.8750613
So is Radius 90°	10.0000000	So Sec. ∠A 33°45'	10.0801536
	11.8750613		11.9552149
	9.8248926		9.8248926
To AB 112.3 Feet	2.0501687	To AC 135 Feet	2.1303223

Pl. II. Fig. 11. 52. *Caſe* 4. Given the Baſe and Hypothenuſe, to find the Angles and Perpendicular.

EXAMPLE. Let the Baſe be 300 Feet, and the Hypothenuſe 500 Feet: Quære the Angles and Perpendicular.

Conſtruction. Make AB = 300 Feet, and raiſe ⊥ BD; then, with an Extent in the Compaſſes = 500 Feet, and one Point in A, with the other deſcribe an Arc, to interſect BD in C; join A, C, and △ ABC is that required.

Calculation. — 1. Making the Hypothenuſe Radius.

53. To find the Angles.		To find the Perpendicular.	
As AC 500 Feet	2.6989700	As Radius 90°	10.0000000
Is to Radius 90°	10.0000000	Is to AC 500 Feet	2.6989700
So is AB 300 Feet	2.4771212	So is S. ∠A 53°7'48"	9.9030895
	12.4771212	To BC 400 Feet	~~1~~2.6020595
	2.6989700		
To S. ∠C 36°52'12"	9.7781512		

54. 2. If the Perpendicular be Radius, there is no Angle given, related to a given Side; and therefore this Caſe cannot be ſolved by making the Perpendicular Radius.

55. 3. Making the Baſe Radius.

As AB 300 Feet	2.4771212	As Radius 90°	10.0000000
Is to Radius	10.0000000	Is to AB 300 Feet	2.4771212
So is AC 500 Feet	2.6989700	So T. ∠A 53°7′48″	10.1249372
	12.6989700	To BC 400 Feet	12.6020584
	2.4771212		
To Sec. ∠ A 53° 7′ 48″	10.2218488		

56, *Scholium*. When the Perpendicular is required, and the Angles are not required, it may be found more expeditiouſly thus. Add the Logarithms of the Sum and Difference of the Hypothenuſe and Baſe together; Half that Sum will be the Logarithm of the Perpendicular. Thus, in the above Example, the Sum of the Hypothenuſe and Baſe = 800; their Difference = 200.

Logarithm of 800 = 2.9030900
Ditto of 200 = 2.3010300

5.2041200

½ 2.6020600 This anſwers in the Table to 400, the ⊥ required.

The Reaſon of this is evident from the Nature of Logarithms, and Art. 305, *Elements of Geometry*.

57. *Caſe* 5. Given the Perpendicular and Hypothenuſe, to find the Angles and Baſe.

Example. Let the Perpendicular be 80 Yards, and the Hypothenuſe 100 Yards: Quære the Angles and Baſe.

Conſtruction. Draw a Line, DB, and at B raiſe the ⊥ BC = 80 Yards; then, with the Extent of the Hypothenuſe, Pl. II. Fig. 12.

64. 3. Making the Baſe Radius.

As AB 120 Feet	2.0791812	As Radius 90°	10.0000000
Is to Radius 90°	10.0000000	Is to AB 120 Feet	2.0791812
So is BC 90 Feet	1.9542425	So Sec. ∠ A 36° 52′ 12″	10.0969105
	11.9542425		
	2.0791812	To AC 150 Feet	12.1760917
To T. ∠ A 36° 52′ 12″	9.8750613		

65. *N. B.* We may find the Angle as above, and then the Hypothenuſe may be found by this Canon. As S. ∠ A : BC :: Radius : the Hypothenuſe.

66. *Scholium.* The common Caſes of right-angled Triangles might have been reduced to a much ſmaller Number, but theſe will not be found too many for the Pupil's Inſtruction.

BOOK II.

Of Oblique Triangles.

67. *Definition.* Every Triangle which has not a Right-Angle is called an Oblique Triangle.

68. Before we proceed to the Caſes of Oblique Triangles it is proper to premiſe a few Theorems, by Way of Lemmata, as the Solutions of the ſeveral Caſes depend on them.

Theorem 1. In all Plane Triangles the Sides are proportional to the Sines of their oppoſite Angles;* *et contra.*

Note,

Pl. II. Fig. 14. * *Demonſtration.* Take AF = BC, and let fall the ⊥s CD, FE; let BC be made Radius, then is alſo AF Radius: Then, by

Note, this is the ſame as is more largely expreſſed in Art 20.

69. *Theorem* 2. If, from the Half Sum of any two Numbers, be taken their Half Difference, the Remainder will be the leſſer Quantity; but the Half Sum, added to the Half Difference, gives the greater Number.†

70.

by right-angled Triangles, FE is the Sine of the Angle A, and CD the Sine of the Angle B. The Lines, FE and CD, being both ⊥ to AB, are parallel to each other; and ∴ the Triangles AEF, ADC, are ſimilar. Hence, as AF : FE :: AC : CD. But AF = BC by the Conſtruction, FE the Sine of the Angle A, and CD the Sine of the Angle B; ∴ the above Analogy becomes, BC : S. ∠ A :: AC : S. ∠ B. Q. E. D.

Corollary 1. Hence it follows, that, if two △s have one ∠ in one = one ∠ in the other, and the Sides oppoſite to thoſe equal Angles are equal, that will have the greateſt Baſe whoſe oppoſite Angle is neareſt to a Right-Angle.

For, by this Theorem, as S. ∠ A : BC :: S. ∠ C : AB. Hence, as A and BC are conſtant, the greater the S. ∠ C is, the greater will AB be; but the S. ∠ C is greateſt when the ∠ C is a Right-Angle; ∴ AB will be greateſt when the ∠ C is a Right-Angle.

Corollary 2. Hence alſo, if the Angles A and C are equal, their oppoſite Sides muſt be equal; and the contrary.

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† *Demonſtration.* Let $2x$ repreſent the greater, and $2y$ the leſſer, Number. Then their Sum is $2x + 2y$, or their Half Sum $x + y$; and their Difference $2x - 2y$, or their Half Difference $x - y$.

To $x+y$	From $x+y$
Add $x-y$	Sub. $x-y$
$2x$. Q. E. D.	Remains $+2y$. Q. E. D.

That x added to x is $2x$, and that y being to be added to x, and ſubtracted from x, will deſtroy each other, and ſo that $2x$ is the Sum of $x+y$ and $x-y$, is ſufficiently clear; but, perhaps, to ſuch as have not learnt Algebra, it may not be ſo evident that $x-y$, ſubtracted from $x+y$, ſhould leave $2y$. If ſo, let it be conſidered that we are only to find the Difference betwixt $x+y$ and $x-y$; now $x+y$ denotes a Quantity, which is y, greater than x; and $x-y$ a Quantity, y, leſs than x; conſequently, as one is y greater, and the other y leſs, their Difference muſt be $2y$.

70. *Corollary.* From hence it follows, that, if from the greater of any two Quantities be taken their Half Sum, the Remainder will be equal to their Half Difference.

71. *Theorem* 3. In any Plane Triangle, as the Sum of any two Sides is to their Difference, ſo is the Tangent of Half the Sum of the two oppoſite Angles to the Tangent of Half their Difference.‡

72. *Theorem* 4. As the leſſer Side is to the greater, ſo is Radius to the Tangent of an Angle: From that Angle ſubtract 45°, and note the Remainder. Then, as Tangent of 45° (or Radius) is to the Tangent of that Remainder, ſo is the Tangent of Half

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Pl. II. Fig. 15. *Geometrically thus.* Let the greater Quantity be repreſented by AC, the leſſer by CB; then is AB their Sum; which biſſect in D; ſo is AD or DB equal to Half their Sum. Take AE = CB; then, as AC is one Quantity, and AE the other, EC is manifeſtly the Difference, and ∴ ED = DC, the Half Difference of the Quantities. Hence it plainly appears, that the Half Sum AD, + the Half Difference DC, is equal to the greater Quantity AC; and that the Half Sum, — DC the Half Difference, is = BC, the leſſer Quantity. Q. E. D.

‡ *Demonſtration.* Let ABC be the Triangle. About C, as a Center, with the Radius CB, deſcribe the Circle EBDE; produce AC to D; and join EB and BD. About E, as a Center, with the Radius EB, deſcribe the Arc B*d*, and with the ſame Radius, about B, as a Center, the Ark E*a*; and draw EF ⊥ to BE. Then is AD = AC + CB, the Sum of the two Sides, and AE = AC — CB, (or CE,) the Difference of the two Sides by the Conſtruction. The ∠ BAC + ∠ ABC = ∠ BCD*. But ∠ BED = $\frac{1}{2}$ ∠ BCD, by Art. 145 of our *Geometry*; ∴ ∠ BED = $\frac{1}{2}$ Sum of the Angles A and ABC. Now, the ∠ EBD, being in a Semicircle, is a Right-Angle, by Art. 148 of our *Geometry*; and ∴ BD, the Tangent of the ∠ CEB, by Art. 7. (Pl. II. Fig. 16.; * 76. g.)

Again: ∠ CBA — ∠ CBE (or CEB) = ∠ EBA, the $\frac{1}{2}$ Difference of the Angles ABC and A, by Art. 70; of which EF is the Tangent. But, by ſimilar Triangles, AD the Sum of the Sides : AE the Difference of the Sides :: BD the Tangent of Half the Sum of the Angles at the Baſe : EF the Tangent of Half their Difference. Q. E. D.

Half the Sum of the two oppoſite Angles to the Tangent of Half their Difference.*

73. This Theorem ſerves for the ſame Purpoſe as the third, only when we have the Logarithms of the Sides given, inſtead of the Natural Numbers; as is the Caſe in calculating the Places of the Planets from aſtronomical Tables, it is Something readier in Practice.

74. *Theorem* 5. In any Plane Triangle, as the Baſe is to the Sum of the other two Sides, ſo is their Difference to the Difference of the Segments of the Baſe.†

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* *Demonſtration.* Let ACb be a right-angled Triangle, right-angled at C, and having two Sides, AC, Cb, reſpectively $=$ the two Sides, AC, CB, of the Triangle ABC. Then, by Theorem 3d, in the right-angled Triangle ACb, As $\mathrm{AC} + \mathrm{C}b : \mathrm{AC} - \mathrm{C}b :: (\mathrm{T}, \frac{\angle \mathrm{A}b\mathrm{C} + \angle \mathrm{CA}b}{2} =$ becauſe the $\angle$ ACb is a Right-Angle) $\mathrm{T}, 45^{\circ} : \mathrm{T}, \frac{\angle \mathrm{A}b\mathrm{C} - \angle \mathrm{CA}b}{2}$ (but $90^{\circ} - \angle \mathrm{A}b\mathrm{C} = \angle \mathrm{CA}b$, $\because$ ſubſtituting for CAb, we get, $\mathrm{T}, \frac{\angle \mathrm{A}b\mathrm{C} - \angle \mathrm{CA}b}{2} = \mathrm{T}, \frac{2\angle \mathrm{A}b\mathrm{C} - 90^{\circ}}{2}) = \mathrm{T}, \overline{\angle \mathrm{A}b\mathrm{C} - 45^{\circ}}$. Alſo, by Theorem 3d, in the Triangle ABC, As $\mathrm{AC} + \mathrm{CB} : \mathrm{AC} - \mathrm{CB} :: \mathrm{T}, \frac{\angle \mathrm{ABC} + \angle \mathrm{CAB}}{2} : \mathrm{T}, \frac{\angle \mathrm{ABC} - \mathrm{CAB}}{2}$; $\because$ by Equality, as $\mathrm{T}, 45^{\circ} : \mathrm{T}, \overline{\angle \mathrm{A}b\mathrm{C} - 45^{\circ}} :: \mathrm{T}, \frac{\angle \mathrm{ABC} + \angle \mathrm{CAB}}{2} : \mathrm{T}, \frac{\angle \mathrm{ABC} - \angle \mathrm{CAB}}{2}$; which is the ſecond Part of the Theorem. Q. E. D. Pl. II. Fig. 16.

Laſtly, by the 6th Caſe of Trigonometry; As bC (CB) : AC :: Radius : Tangent of $\angle$ AbC; which is the 1ſt Part of the Theorem. Q. E. D.

† *Demonſtration.* Let ABC be the Triangle, AB the Baſe. About C, as a Center, with the Radius CB, deſcribe the Circle Pl. II. Fig. 17.

75. *Corollary*. This Theorem gives alſo twice the Diſtance of the Perpendicular from the Middle of the Baſe.

For, if we put *m* to repreſent the Middle of the Baſe, and *b* Half the Baſe; then $AE = b + mE$, and $BE = b - mE$; and, conſequently, the Difference of the Segments $= 2mE$.

76. The common Caſes of oblique Triangles are,

Caſe 1ſt. Given two Angles and one Side, to find the other Sides. This is ſolved by *Theorem* 1.

Pl. III. Fig. 1. **Example.** Let the Baſe AB be 150 Yards, the ∠ A 30 d. and ∠ B 40 d. Quære AC and BC.

Conſtruction. Make AB = 150 Yards, ∠ A = 30 d. ∠ B = 40 d. and draw AC and BC to interſect in C; then is the Triangle ABC that required.

Calculation. The ∠ A 30 d. + ∠ B 40 d. = 70 d. and 180 d. − 70 d. = 110 d. = ∠ C.

To find BC.		To find AC.	
As S. ∠ C 110°	9.9729858	As Sup. ∠ C 70°	9.9729858
Is to AB 150 Yards	2.1760913	Is to AB 150 Yards	2.1760913
So is S. ∠ A 30°	9.6989700	So is S. ∠ B 40°	9.8080675
	11.8750613		11.9841588
	9.9729858		9.9729858
To BC 79.81 Yards	1.9020755	To AC 102.6 Yards	2.0111730

77.

cle GFBDG; join C, F; biſect FB by the Line CE; then, CB being = CF, FE = EB, by Conſtruction, and CE common, the Triangles FEC and BEC are equal in every Reſpect, by our *Geometry*, Art. 95. Therefore ∠ FEC = ∠ CEB; or, in other Words, CE ⊥ to AB. The two Parts, into which the Baſe is cut by the ⊥ CE, are called *Segments*.

In our Eſſay on Geometry, Art. 212, it is demonſtrated, that AB × AF = AD × AG. Hence this Analogy, As AB : AD :: AG : AF. But AB is the Baſe; and it is plain, from the Conſtruction, that AD is = the Sum of the Sides AC and CB; AG = their Difference, *viz.* = AC — CG (or CB); and AF = AE — EF (or EB) = the Difference of the Segments.

The above Analogy, expreſſed in Words, will be as in the Theorem.

77. *Caſe* 2. Given two Sideſ, and an Angle oppoſite to one of them, to find the other Side and Angles.

EXAMPLE. Let AB = 82 Feet, AC = 66 Feet, the ∠ B = 49 d. 30′; quære the Side BC, and Angles A and C. Pl. III. Fig. 2.

Conſtruction. Make AB = 82 Feet; ∠ B = 49 d. 30′: then, taking a Diſtance of 66 Feet between the Compaſſes, place one Leg on A, and turn the other about to interſect the Line BD, which it will do in two Places, C and E; ſo that this Caſe admits of two Solutions; for either of the Triangles, ABC or ABE, has the Conditions required in the Queſtion, and therefore this is called the doubtful Caſe, unleſs it is known, from ſome Circumſtance, whether the Angle oppoſite AB is obtuſe or acute. If acute, Theorem 1ſt finds the Angle E; if obtuſe, its Supplement to 180 d. is the Angle C: for it has been already obſerved, that any Arc and its Supplement to 180 Degrees have the ſame Sine.

In the following Solution we will ſuppoſe the Angle oppoſite to AB to be obtuſe, and ſo ABC to be our Triangle.

CANONS.

To find the ∠s C and A.	To find BC.
As AC 66 Feet 1.8195439	As S. ∠ B 49° 30′ 9.8810455
Is to S. ∠B 49°30′ 9.8810455	Is to AC 66 Feet 1.8195439
So is AB 82 Feet 1.9138139	So is S. ∠ A 21° 22′ 9.5615010
To Sup. ∠C 70°52′ 11.7948594	11.3810449
	9.8810455
Hence ∠C = 180° — 70° 52′ = 109° 8′; and ∠ A — 180° — 109° 8′ — 49° 30′ = 21° 22.	To CB 31.62 Feet 1.4999994

78. *Caſe* 3. Given two Sides and the Angle contained between them, to find the other Angles and Side.

The Angles are found by Theorems 3d and 2d; then the other Side by Theorem 1ſt.

Pl. III. Fig. 3. EXAMPLE. Let the Angle A = 22 d. 30′, AC = 300 Yards, and AB = 400 Yards: Quære the Angles at B and C, and Side BC.

Conſtruction. Make AB = 400 Yards, ∠ A = 22 d. 30′, AC = 300 Yards; join B, C; and the Triangle ABC is that required.

			d.	′
To and from AB	400	From	180	0
Add and ſubtract AC	300	Subtract ∠ A	22	30
Gives Sum of Sides	700	Gives ∠B + ∠C	157	30
Their Difference	100	Half Sum of the ∠s	78	45

CANONS.

To find the Angles.		To find BC.	
As Sum of Sides 700 Yards .	2.8450980	As S. ∠B 43° 4′	9.8343246
Is to their Dif. 100 Y.	2.0000000	Is to AC 300 Yards	2.4771213
So is Tan. ½ Sum unknown ∠s 78° 45′ .	10.7013382	So is S. ∠A 22° 30′	9.5828397
	12.7013382		12.0599610
	2.8450980		9.8343246
To Tan. of ½ their Diff. 35° 41′	9.8562402	To BC 168.12 Yds	2.2256364
To and from	78° 45′		
Add and ſubtract	35 41		
Gives ∠C	114 26		
∠B	43 4		

This Caſe may alſo be ſolved by Theorem 4th, and by letting fall a Perpendicular, as in Pl. III.

Fig. 4, independently of either of theſe Theorems; for in the right-angled Triangle ADC will be given the Hypothenuſe AC, and Angle A, to find, by Right-Angle Trigonometry, the Baſe AD, and Perpendicular DC. Then AB — AD = BD; and therefore, in the right-angled Triangle BDC, are given BD and DC, to find the Hypothenuſe BC.

79. *Caſe* 4. Given the three Sides, to find the Angles.

Pl. III. Fig. 4.

Method of Calculation. From the greater Angle let fall the Perpendicular DC; with the Radius CD deſcribe the Arc BE; then are BD and AD called Segments; and, BD being = DE, AE is the Difference of the Segments, which is found by Theorem 4th: Then from AB ſubtract AE, and the Remainder is = EB; Half of which is ED, or its Equal DB, the ſhort Segment.

Now, the Triangle is divided into two right-angled Triangles; in each of which are given the Hypothenuſe and Baſe, to find the Angles, by Caſe 4th of right-angled Triangles, or by Theorem 1ſt. Laſtly, the ∠ACD, added to the ∠DCB, will give the whole ∠ACB; or the ∠s A and B, added together, and ſubtracted from 180, are = the ∠ACB.

Example. Given AB = 350 Feet, AC = 240 Feet, BC = 200 Feet; to find the Angles.

The Triangle is conſtructed by the Method ſhewn in Art. 224. of the *Eſſay on Geometry*.

The Operation by the above Directions will be as under.

AC + BC = 440, and AC — BC = 40.

CANONS.

CANONS.

To find the Difference of the Segments.

As AB 350 Feet	2.4540680
Is to AC+BC 440F.	2.6434527
So is AC—BC 40F.	1.6020600
	4.2455127
	2.5440680
To AE Diff. of Segments, 50.29 Feet .	1.7014447

From AB =	350
Subtract AE =	50.29
EB =	299.71
ED or DB ½ the short Segment . . .	149.85
Add AE =	50.29
Gives AD long Segment	200.14

To find the Angles A and ACD.

As AC 240 Feet	2.3802112
Is to Radius 90°	10.0000000
So is AD 200.14 Feet	2.3013339
	12.3013339
	2.3802112
To S. ∠ ACD 56° 30′ .	9.9211227

And 90° — 56° 30′ (∠ACD) = 33° 30′ = the ∠ A.

To find the Angles B and BCD.

As CB 200 Feet	2.3010300
Is to Radius 90°	10.0000000
So is DB 149.85 Feet	2.1756567
	12.1756567
	2.3010300
To S. ∠ DCB 48° 32′ . .	9.8746267

And 90° — 48° 32′ (∠DCB) = 41° 28′ = the ∠ B.

To find the Angle C.

The ∠ A 33° 30′ + ∠ B 41° 28′ = 74° 58′; and 180° — 74° 58′ = 105° 2′ = ∠ C.

Or, ∠ACD 56° 30′ + ∠DCB 48° 32′ = 105° 2′ = ∠ C.

BOOK

BOOK III.

Of the improved Navigation-Scale.

80. HAVING, in many Years Teaching, obſerved, that the *Navigation-Scale*, commonly called, by Sailors, *Gunter's Scale*, was generally inaccurately made, and very deficient in Lines of equal Parts, I long wiſhed for an Improvement thereon; and, in the Year 1764, being in *London*, I took that Opportunity to have ſome more accurately made, with ſome Improvements of my own; and, in the Beginning of the Year 1772, being again in Town, I readily embraced that Opportunity to make ſome farther Improvements, and a new Pattern, with the Aſſiſtance of a good Workman; ſo that I flatter myſelf, if compared with the common Scales, they will be found more generally uſeful and accurate. Several incorrect pirated Copies, made in a very unworkmanlike Manner, and ſome with even the Diagonals drawn wrong, having been ſold in *London* and *Briſtol*, and perhaps in ſome other Places, as my improved Scales; to prevent as much as poſſible my own Character from being injured thereby, and the Public from being impoſed on, I hereby deſire that any of the principal Shops, inclined to ſell them, will apply to me to know with whom I have left the Pattern for the Trade, and to put, under *Improved by B. Donn*, their own *Names* as *Makers*: and the Public may be aſſured, that, whoever ſhall offer to Sale the Scale without my Pamphlet of its Uſe, is ſelling a pirated Copy of the Scale, and therefore not to be depended on.

81. The ſeveral Particulars, in which this Scale, as now improved, excels the common Scales, will plainly appear, from the following Deſcription and

Uſe, to thoſe who will take the Trouble to compare it with the common Scales.

82. The Plane or Plotting Side, *viz.* that which has the Bevil Edge, contains the following Scales.

1. A Scale of Inches, divided into *Tenths* of an Inch. On the Left, or at the Beginning of the Inches, are ſeveral Braſs Pins, with Figures annexed, which ſhew the Weight of the Ball to any Diameter of the Gun. For Example: If the Diameter of the Bore of a Gun be 5 Inches and 4 Tenths, it ſhews by Inſpection that ſuch a Bore will carry 18 Pound Shot. This Addition is made, as it may be uſeful to Numbers of Sailors; for whoſe Uſe it may be proper to hint, that the Weight of Powder for Service is generally about Half the Weight of the Shot.

2. Under the Line of Inches, a Foot is divided into 100 equal Parts, ſo as by Inſpection to turn readily Inches into the Decimal of a Foot, or the contrary. Its Uſe is known to ſuch as are acquainted with Menſuration. Dimenſions may alſo be taken in Feet and decimal Parts, inſtead of Feet and Inches, *&c.* and ſo the Content of any Piece of Timber, *&c.* found by common Multiplication. For Example: Suppoſe a Plank is 20 Feet long, and 1 Foot 50 Hundredths broad; then 1.50, multiplied by 20, gives 30 Feet, the Content required. *This Line is not on the common Scales.*

3. The common Scales contain but few Scales of equal Parts; by which Means the young Navigator and Surveyor was frequently at a Loſs for a proper Scale to conſtruct his Scheme with. To remedy this, I have ſeen ſome Scales made, by the Direction of an ingenious Gentleman, which contained more Scales of equal Parts than the common ones; but, in Order to make Room for them, the Diagonal Scales were omitted. But, in Caſes which require particular Accuracy in conſtructing, the Diagonal Scales are neceſſary; there-

fore I have contrived to give *even two more Scales, and yet retained the Diagonal-Scales,* by diſpoſing of the Lines in different Order. The Scales of equal Parts are 10, 15, 20, 25, 30, 35, 40, 45, 50, and 60, to an Inch.

4. This Side contains likewiſe Lines of Rumbs, Chords, Sines, Tangents, *&c.* as on the common Scales; alſo two Lines, correſponding to each other, one marked *M. Lon.* the other *Chord*, by which are ſhewn, by Inſpection, how many Miles make a Degree of Longitude in any Latitude. For Example, againſt 60 Degrees of Longitude, on the *Chord*, you will find, on the *M. Lon.* that 30 Miles make a Degree of Longitude in that Latitude.

I ſhall now proceed to deſcribe the Contents of the other Side, *viz.* that which is commonly called *Gunter's*, from the Logarithms of Numbers, Sines, and Tangents, being firſt laid thereon by Mr. *Edward Gunter*.

83. For the Sake of the young Student, it may be proper to obſerve, that, to work a Canon or Proportion on *Gunter*'s Scale, we have only to extend the Compaſſes from the firſt Term to the ſecond, on the proper Line, then will the ſame Extent, laid the ſame Way from the third Term, (*viz.* from the Left to the Right, or from the Right to the Left, according as the ſecond Term lay from the firſt,) give, on its proper Line, the fourth or required Number.

84. As, in the common Rule of Three, it is no Matter which of the two middle Terms is placed firſt, ſince the Product is the ſame, ſo, in working on the *Gunter*'s Scale, if at any Time it is found more convenient, we may extend the Compaſſes from the firſt Term of the Proportion to the third Term, then will that Diſtance extend from the ſecond Term to the fourth or required Term.

85. In counting on the Line of Numbers, it may be proper to obſerve, that the Numbers 1, 2, 3, 4, *&c.* repreſent not only 1, 2, 3, 4, *&c.* but 10, 20,

30, &c. Thus, if the Canon you are working of requires you to call the 1, in the Middle of the Line of Numbers, 100, then muſt the 2, 3, 4, 5, 6, 7, 8, 9, 10, on the Right-Hand, be called 200, 300, 400, 500, 600, 700, 800, 900, 1000; the 9, 8, 7, 6, 5, 4, 3, 2, 1, on the Left, be called 90, 80, 70, 60, 50, 40, 30, 20, 10.

If the 1 in the Middle be called 1000, the Figures 2, 3, 4, &c. on the Right, will be 2000, 3000, 4000, &c. and 9, 8, 7, 6, &c. on the Left, 900, 800, 700, 600, &c.

Again, if the 1, in the Middle of the Line of Numbers, be called 10, the Figures 2, 3, 4, &c. on the Right-Hand, muſt be 20, 30, 40, &c. and, on the Left, 9, 8, 7, &c. will be only 9, 8, 7, &c.

Alſo, if the 1 in the Middle be called 1, then 2, 3, 4, &c. on the Right-Hand, will ſtand for 2, 3, 4, &c. but 9, 8, 7, &c. on the Left, will be only $\frac{9}{10}$, $\frac{8}{10}$, $\frac{7}{10}$, &c. &c.

86. For the young Student, the Canons, in common Uſe in working a Day's Work, are ſtamped on the Scale. Thus R : Diſt. :: SC : Dep. :: SCC : D. Lat. ſignifies, that, as Radius is to the Diſtance, ſo is Sine of the Courſe to the Departure, and ſo is Sine-Complement of the Courſe to the Difference of Latitude.

Again, D. Lat. : Dep. :: Tan. 45 : T. Courſe, is, as Difference of Latitude is to the Departure, ſo is Tangent of 45 Degrees to the Tangent of the Courſe.

Alſo, SC mid. Lat. : Dep. :: R : D. Long. is to be read, as Sine-Complement of middle Latitude is to the Departure, ſo is Radius 90 Degrees to the Difference of Longitude. Or, to find the Difference of Longitude by meridional Parts, D. Lat. : Dep. :: M.D. Lat. : D. Long. which is to be read, as Difference of Latitude is to the Departure, ſo is meridional Difference of Latitude to the Difference of Longitude.

Laſtly, SCL : S⊙Dec. :: R : S⊙Amp. is to be read, as Sine-Complement of Latitude is to the Sine of the Sun's Declination, ſo is Radius, or Sine of 90 Degrees, to the Sine of the Sun's Amplitude.

87. The *firſt Line* on the *Gunter*'s Side is marked *S. Rumb*, that is, Sines of Rumbs or Courſes.

88. The *ſecond Line* is marked *Numb. Sqr.* and is the common Line of Numbers intended to be uſed with the Sines or Tangents, *&c.* in working the uſual Canons in Trigonometry or Navigation. It is marked *Numb. Sqr.* becauſe, with the Line marked *Root*, it will ſerve for ſquaring Numbers, *&c.* of which farther Notice will be taken preſently.

89. The *third Line*, marked S | CS | Sec. is the common Line of Sines, with the Addition of being numbered backward by ſmall Figures; by which Addition it now ſerves the ſeveral Purpoſes of a Line of *Sines*, *Co-ſines*, and *Secants*. A Notion has pretty generally prevailed, amongſt Teachers of Trigonometry and Navigation, that the Canons, in which Secants are concerned, could not be ſolved by *Gunter*'s Scale; and alſo amongſt ſome Writers on Navigation; for Mr. K——y, in his Navigator's Tutor, and Mr. M——y. in his Rudiments of Navigation, both poſitively affirm, that, *If there is a Secant in the Proportion, the Operation cannot be performed by the Scale.* But that they are miſtaken may be readily ſhewn, by only working one Example. Let us ſuppoſe the Angle A, or Angle at the Baſe of a Rt. ∠ △ to be given, equal to 50 Degrees, and the Baſe 86 Yards, to find the Hypothenuſe. If the Baſe be made Radius, the Canon is, as Radius is to the Baſe 86, ſo is Secant-Angle, A 50 Degrees, to the Hypothenuſe.

This may be worked on the Scale thus: Extend the Compaſſes from Radius, or Sign of 90 Degrees, to 50 Degrees, on the ſame Line, counting backward from Radius, by the ſmall Figures, for the Secant; then, one Leg of the Compaſſes being ſet at

86,

86, on the Line of Numbers, that Extent will, turned the contrary Way, (becauſe the Secants muſt be conceived to run beyond the Scale,) reach to 134, the Diſtance which was required.

90. The *fourth* and *fifth* Lines are the verſed Sines and Tangents, as on common Scales.

91. The *ſixth* is the Meridian-Line, which, on the common Scale, is too ſhort to be of much Service, but, by making it on two Lines, we are enabled to go to 80 Degrees of Latitude, and yet have the Degrees ſufficiently large to divide the Meridian of WRIGHT's Chart into every 10 Minutes, or leſs Parts. — The Length of the Degree of the Equator, correſponding to theſe of the Meridian, is annexed on the Left-Hand, and marked *Eq. Deg.*

92. The Meridian-Lines being conſtructed by the diagonal Scale of 20, if, at any Time, in the Abſence of Tables, we are inclined to know nearly the Meridional-Difference of Latitude between any two Places, we have only to take the Diſtance between the two Latitudes from the Meridian-Line, then, meaſuring the Diſtance on the diagonal Scale of 20, we ſhall have nearly the meridional Miles required. Thus, for Example, if it were required to find the meridional Diff. of Lat. between the Latitudes of 20 and 30 Degrees, the Diſtance between theſe Degrees, taken from the Meridian-Line, will, on the leſſer diagonal Scale, meaſure 663 (reckoning each primary Diviſion, or Half-Inch, 100 Miles) for the Meridional-Difference of Latitude.

93. The *ſeventh* and *eighth*, or two remaining Lines of this Scale, are a ſingle and triple Line of Numbers, marked *Numb. Root*, *Numb. Cube*, which, together with the ſecond Line, or double Line of Numbers, marked *Numb. Sqr.* are for *ſquaring* or *cubing* Numbers, or for extracting the *Square* and *Cube Root*, or for working *Proportions wherein Squares and Cubes, or the Square and Cube Roots, are concerned*, of

great Uſe in various Parts of the Mathematics. We ſhall give a few Examples of their Uſe.

94. EXAMPLE 1. What is the Square-Root of 144? or, which is the ſame Thing, what is the Side of a Square whoſe Area is 144 Feet?

Solution. Call the 1, at the Beginning of the Line of Numbers, (marked *Numb. Sqr.*) 100, and extend the Compaſſes from that Point to 144; then, calling the 1, at the Beginning of the Line of Numbers, (marked *Numb. Root,*) 10, and putting one Point of the Compaſſes in that Point, the other will extend to 12, the required Root.

95. EXAMPLE 2. *The Areas of Circles being as the Squares of their Circumferences,* let us ſuppoſe the Weight of Cables, of unequal Circumferences but of equal Lengths, to be in the ſame Proportion. On this Suppoſition, let it be required to find the Weight of a Cable whoſe Circumference is 8 Inches; the Weight of one, of equal Length, of 10 Inches Circumference, being 25 Hundred. Here, by the Suppoſition, the Proportion is, as the Square of 10 is to the Square of 8, ſo is 25 to the required Weight.

To ſolve this on the Scale, extend the Compaſſes, on the Line *Numb. Root,* from 10 to 8, then, on the Line *Numb. Sqr.* will that Extent reach from 25 to 16 Hundred, the Weight of the Cable of 8 Inches.

96. EXAMPLE 3. What is the Cube-Root of 1728? or, which is the ſame Thing, if a Cube or Die contains 1728 ſolid Inches, what is the Side of the Cube?

This may be ſolved by a bare Inſpection; for, calling the 1, at the Beginning of the Line *Numb. Cube,* 1000, and the 1 correſponding to it, on the Line *Numb. Root,* 10, then, againſt 1728, on *Numb. Cube,* you will ſee 12, the required Root, on *Numb. Root.*

97. EXAMPLE 4. Suppoſe a Ship of 300 Tons to be 75 Feet by the Keel, it is required to find the

Length of the Keel of a ſimilar Ship of 500 Tons Burthen.

Solution. Similar Solids being in Proportion as the Cubes of their like Sides, we have, as 300 is to 500, ſo is the Cube of 75 to the Cube of the required Keel. Therefore, extend the Compaſſes, on the Line *Numb. Cube*, from 300 to 500, (or from 3 to 5,) then will that Extent, on the Line *Numb. Root*, reach from 75 to 89 Feet nearly, the Length of the Keel required. This Inſtance of a uſeful but troubleſome Queſtion, if performed by Arithmetic, being ſo expeditiouſly ſolved on the Scale, by the Addition of theſe Lines, is a ſufficient Recommendation of their Uſe.

98. What we have ſaid on the Uſe of the improved Scale is ſufficient for thoſe who are acquainted with the Uſe of the common Scales; as for others, it would be adviſeable for them to apply to a proper Maſter for Inſtructions. However, I intend to treat more fully of the Uſes of the ſeveral Lines, with their Conſtructions, in my Treatiſe of Navigation, to which the Subject more naturally belongs.

99. To prevent the Public being impoſed on, by the Sale of pirated Scales, it may be proper to acquaint them that I have appointed the following Mathematical-Inſtrument-Makers, in *London*, to vend them, *viz.* Meſſieurs *Adams*, *Heath* and *Wing*, *Lincoln*, *Martin*, *Nairne*, *Watkins*, and *Whitford*: As they will not ſell any without the Pamphlet of their Uſe, and their own Names, as Makers, ſtamped on the Scales, conſequently, their own Reputation will not permit them to ſell any made in an inaccurate or unworkmanlike Manner.

BOOK

BOOK IV. *Of Altimetry and Longimetry.*

CHAP. I. *Of Altimetry.*

100. ALTIMETRY is the Application of the Doctrine of Triangles to find the Heights of Objects, whether acceſſible or not.

101. If an Object ſtands on a Plane, ſo that a Line may be meaſured from the Obſerver to the Foot of the Object, it is ſaid to be acceſſible, otherwiſe not.

102. *Prop.* 1. To find the Height of an acceſſible Object.

This may be done ſeveral Ways, as in the following Caſes.

Caſe 1. To find the Height, when the Sun (or Moon) ſhines.

EXAMPLE. Let it be required to find the Height of a Tower, the Length of whoſe Shadow, on the horizontal Plane, was 90 Feet, at a Time when the Shadow of an upright Pole, of 10 Feet Height, was 9 Feet. Pl. III. Fig. 5.

Conſtruction. Let AB repreſent the Plane; make AD = 9 Feet, and $\perp$ DE = 10 Feet: Alſo AB = 90 Feet; and raiſe the $\perp$ BF; produce AE to interſect it in C; then it is plain BC muſt be the Height of the required Tower.

Solution. It is manifeſt, the △s ADE, ABC, are ſimilar: ∵ as AD, the Shadow of the Pole, : DE, the Height of the Pole, :: AB, the Shadow of the Tower, : BC, the Height of the Tower. In this Example, as 9 : 10 :: 90 : 100, the Height of the required Tower.

103. *Caſe* 2. To find the Height, by a Walking-Stick or Pole, when the Sun does not ſhine.

EXAMPLE. Suppoſe I want to find the Height of the Tree CE, above the horizontal Plane AC, by Means of a Pole of 15 Feet in Length.

Pl. III. Stick the Pole BF $\perp$ to the Plane AC, at a con-
Fig. 6. venient Place; then walk backwards from it to A, till you ſee the Points F and E in a right Line: This done, meaſure from A to B, alſo from A to C. Let us ſuppoſe AB = 10 Feet, AC = 20 Feet, and the Height of the Obſerver's Eye, AH, = 5 Feet.

Conſtruction. Make AB = 10 Feet and the $\perp$ BF = 15, alſo AC = 20 Feet, and erect $\perp$ CI, which interſect by HF produced in E; then is CE evidently = the Height of the Tree required.

Solution. Draw HD $\parallel$ to AC; then it is manifeſt the Δs HGF and HDE are ſimilar. Therefore, as HG (AB) : GF (BF−BG) :: HD (AC) : DE. In our Example, as 10 : 10 :: 20 : 20 = DE. Then DE+DC (AH) = 20+5=25 Feet, the Height of the Tree required.

104. *Caſe* 3. To find the Height by Means of a Bowl of Water, or other horizontal reflecting Surface.

Method. Let A repreſent the Plane of the reflecting Surface, DB the horizontal Plane: Walk backwards
Pl. III. to D, till your Eye, at E, ſees the Reflection of the
Fig. 7. Top of the Object BC in the Baſon A; then meaſure DA, which ſuppoſe = 7 Feet, and AB = 70 Feet, the Height of the Eye, DE, = 5 Feet.

Conſtruction. Make AD = 7 Feet, AB = 70 Feet, and erect $\perp$s. Make DE = 5 Feet, and $\angle$ BAC = $\angle$ DAE, then will BC be the Height of the Object required; becauſe, by the Nature of Reflection, the $\angle$s DAE and BAC muſt be always = each other. See Art. 317 in our Eſſay on Geometry.

Solution. The $\angle$s ADE, ABC, being manifeſtly = each other, we have, as AD : DE :: AB : BC, that is, in the above Example, as 7 : 5 :: 70 : 50 = BC, the Height of the Object required.

105. *Caſe* 4. To find the Height, by Means of a Quadrant or other proper Inſtrument to take the Angle of Elevation, of the Top of the Object.

EXAMPLE. Suppoſe, ſtanding at A, the Eye at D, the Angle of Elevation EDC is taken = 40 Degrees;

and from A to B the Foot of the Object meaſures 100 Feet: Quære the Height of the Object BC, the Eye D being 5 Feet above the horizontal Plane AB.

Conſtruction. Make AB = 100 Feet, the ⊥ AD = 5 Feet; alſo erect the ⊥ BF, and draw DC, making an Angle EDC, = 40 d Then is BC evidently = the Height of the Object required. Pl. III. Fig. 8.

Solution. By Trigonometry, as Coſine ∠ D, or Sine ∠ C, : DE=AB :: S∠D : EC: Then EC +EB (DA) = BC, the required Height.

Operation by Logarithms.

As Coſine 40 d.	9.8842540	
Is to 100 Feet	2.0000000	
So is Sine 40 d.	9.8080675	EC = 83.91 Feet
	11.8080675	
	9.8842540	EB = 5 Feet
To EC 83.91 Feet	1.9238135	BC = 88.91 Feet

106. *Scholium.* As, perhaps, ſome of our Readers may not be acquainted with the Nature of a Quadrant, it may be proper to deſcribe that commonly made Uſe of in Altimetry.

Conſtruction. On a Piece of Braſs, or dry Board that will not eaſily warp, make a Quadrant, or Quarter of a Circle, ABC. Draw *ab* || to AB. About *a*, as a Center, deſcribe the Arc *bc*; and, ſince the Radius is = the Chord of 60 Degrees, with the ſame Extent, and one Point of the Compaſſes in *b*, with the other, make a Mark in the Arc at *e*; then biſect the Arc *be* in *d*; take the Diſtance *bd* and ſet from *e* to *c* and join *ac*; then is *bd*=*de*=*ec*=30 Degrees; and therefore the whole Arc *bdcc* = 90 Degrees, or a Quarter of a Circle. See the F. in Pl. IV.

By a Trial or two, divide the Arc *bd*, *de*, and *ec*, each into 3 equal Parts, or every 10 Degrees; then biſect each of theſe into every 5 Degrees; laſtly, by a few Trials, into every Degree; which, if the Quadrant be ſufficiently large, as a Foot Radius, may be ſubdivided into Halves, Quarters, or leſſer Diviſions, *viz.* into every Minute, by Diagonals, as will be explained when we deſcribe the Quadrants uſed at Sea. Make $\angle baf = \angle fac = 45$ d. draw $fb \parallel ag$ and $gf \parallel ab$, then will *agfb* be a Square, by the Conſtruction. Divide *gf*, *fb*, each into 100 or 1000 equal Parts. From the Center *a* hang a Plumb-Line *a*D; the finer the Line is the better, a †Horſe-Hair will do very well; the Weight may be either Lead or Braſs; and, if the Line be hung by a little Bow, on a Pin at *a*, inſtead of being run through a Hole, it will play better.

It only remains to fix two convenient Sights, *ik*, on the Edge of the Quadrant, AC, or on any Line $\parallel$ thereto. Theſe are commonly made only of two Bits of Braſs, with a little Hole in each, of the ſame Height: But as, by theſe Means, it is very difficult to ſee the Object through the fartheſt Sight, by Means of the ſmall Aperture or Opening of the Hole, I generally make them in the Manner repreſented by the Figures *i* and *k* in the Plate. The large Apertures are made ſufficiently open to take in a large Field of View, and have Croſs-Hairs or fine Silver Wires interſecting each other at right $\angle$s, ſo that each Interſection is exactly of the ſame Height as the Hole in the Sight next the Eye.

If the Quadrant be too large to hold ſteady by Hand, it may be fixed to a Ball and Socket, and ſupported by the Legs of a common Surveying-Table, (generally called a Plain-Table,) or to a Pole like that of a Barber's Block.

107. The Manner of uſing this Quadrant, to take an Angle of Altitude of any Object, is thus:

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† Or fine Silver Wire.

Holding the Quadrant upright in your Hands, look through the Sights to the Top of the Object G, then will the Number of Degrees, cut by the Plumb-Line, *viz.* the Arc GH, expreſs the Angle of Altitude. Pl. III. Fig. 9

Demonſtration. Draw DE $\parallel$ AB. Then, DE being $\parallel$ to the Horizon, by the Nature of a Plumb, the $\angle$DIF muſt be a Right $\angle$, and ſo the $\triangle$ DIF a right-angled one; therefore, (by 83 of *Geometry*,) the $\angle$EDF (IDF) $+$ $\angle$HFD (IFD) $=$ a Right $\angle$; and $\angle$ GFD being alſo a Right-Angle, the $\angle$ GFH $+$ $\angle$HFD $=$ a Right $\angle$. Therefore, taking away the $\angle$ HFD, common to both, the $\angle$ EDF (the Angle of Elevation) is $=$ $\angle$GFH $=$ the Number of Degrees in the Arc GH. *Q. E. D.*

The Angle of Depreſſion of an Object, L, *viz.* the $\angle$LFK, (FK being $\parallel$ BA,) is found in nearly the ſame Manner, by only placing the Eye at the Center of the Quadrant, F, (inſtead of at D,) and looking through the Sights FD to L. Then is the $\angle$ GFH $=$ the $\angle$ of Depreſſion, LFK.

108. *Caſe* 5. To find the Height of an acceſſible Object by Means of the Quadrat, which is of great Uſe when abſent from Tables or proper Inſtruments for calculating

The Method is this. Look at the Top of the Object through the Sights of the Quadrant, as before explained; then, inſtead of looking the Degrees on the Arc, count the Diviſions between *c* and *d* cut by the Plumb-Line; which reſerve. Conceive a Line, DE, $\parallel$ to AB, and BC $\perp$ to both: Then, the $\angle$s DEC, *dcb*, being Right $\angle$s, and $\angle$ *cbd* $=$ $\angle$ of Elevation, EDC, (as has been already ſhewn, in the Uſe of the Quadrant,) it follows, *cdb* muſt be $=$ $\angle$DCE, by Art. 83 *Geometry*. Hence the $\triangle$s *bcd*, DEC, are ſimilar; $\because$ As *bc* : *cd* :: DE : EC. This is when the $\angle$ of Elevation is leſs than 45 d. or the Plumb falls between *c* and *e*, or on the Side of the Quadrat called Siniſtra (or Left-Hand). But, Pl. III. Fig. 10.

when the ∠ of Elevation is more than 45 d. or the Plumb-Line falls between *a* and *e*, on the Side called Dextra, or Right-Hand, then, by ſimilar Triangles, as *ad* : *ab* :: DE : EC.

Pl. III. Fig. 11.

109. Example. Standing at A, my Eye being at D, 5 Feet above the horizontal Plane AB, I found, by the Quadrat, that *bc* being divided into 1000 Parts, *cd* was = 500; and from A to the Middle of the Baſe of the Obeliſk meaſured 80 Feet. Quære the Height of the Obeliſk.

Pl. III. Fig. 10.

By the above, as *cb* : *cd* :: DE (AB) : EC. ∴ in this Example, by the Golden-Rule, As 1000 : 500 : 80 : 40 Feet.

To EC = 40 Feet,
Add BE = 5 = AD, the Height of the Eye.

Gives 45 Feet, the Height required.

110. *Prop.* 2. To find the Height of an inacceſſible Object, CE, ſtanding erect on an horizontal Plane.

Caſe 1. By the Quadrant.

Suppoſing a River, Ditch, or ſome other Obſtruction, between the Foot of the Object and the Obſerver, ſo that the Diſtance BC cannot be meaſured on the Ground: Therefore, at B he can only take the ∠ DGE 50 d. and, walking backwards, meaſure a convenient Diſtance, BA, (ſuppoſe 67 Feet,) and then take the ∠ of Elevation, DFE = 35 d. Hence it is required to find the Height of the Object CE.

Pl. III. Fig. 12.

Conſtruction. Make a Line FG = 67 Feet, and produce it towards D at Pleaſure; make ∠ DGE = 50 d. ∠ DFE = 35 d. then will the Point of Interſection be the Top of the Object; and, letting fall the ⊥ DE, produce it to C, ſo that DC be made = the Height of the Eye, which we ſuppoſe 5 Feet; then it is manifeſt CE is the Height required.

Calculation. Firſt, 180 d.—∠DGE = ∠FGE, in our Example, = 130 d. and ∠DFE + ∠FGE = 165 d. which, ſubtracted from 180 d. gives ∠FEG = 15 d. Hence, by Trig. as S.∠FEG 15 d. : FG 67 Feet :: S.∠ GFE 35 d. : GE = 148.5 Feet. Now, in the Right∠d △ GDE, we have, as Radius 90 d. : GE = 148.5 Feet :: S.∠DGE = 50 d. : DE = 113.7 Feet; to which adding DC = 5 Feet, we have CE = the Height of the Object required, = 118.7 Feet.

If the young Student is inclined, he may find the Side EF by ſaying, as S.∠FEG : FG :: S.∠FGE : FE. Then, as Radius : FE :: S.∠DFE : DE. Which coming out the ſame as before will be a Proof of the whole Work.

If BC be required, As Radius : GE :: Coſine ∠DGE : GD = BC.

111. *Caſe* 2. But, if no *Tables* are at Hand, then the Height may be found by the *Quadrat.*

Thus, ſuppoſe the Plumb cuts, on the Dextra-Side, 845 at B, and at A, on the Siniſtra-Side, 700, counting each Side of the Quadrat 1000, and AB 67 Feet. Quære the Height.

Rule. * From the Square of the Number, into which each Side of the Quadrat is divided, (that is, from 10000, if the Side is divided into 100 Parts, or from 1000000, if into 1000 Parts,) ſubtract the Product of the Numbers cut by the Plumb, on both the Dextra and Siniſtra Sides, and reſerve the Remainder. Laſtly, divide the continued Product of the Numbers cut on the Dextra-Side, the Siniſtra-Side, and Length FG, by the reſerved Number; the Quotient will be = to GD: Then, by Propoſition 1, Caſe 5, find DE.

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* *Demonſtration.* Let a = each Side of the Quadrat, s = the Number cut by the Plumb-Line on the Siniſtra-Side, d = the Number on the Dextra-Side, m = FG, x = GD; then FD = FG + GD = $m+x$. Now, by Propoſition 1, Caſe 5, As $a : s$ $:: m+x : \frac{sm+sx}{a}$ = DE: Alſo, As $d : a :: x : \frac{ax}{d}$ = DE: Therefore, $\frac{ax}{d} = \frac{sm+sx}{a}$. Hence, $a^2x = smd + sdx$. And ∴ $x = \frac{smd}{a^2 - sd}$. *Q. E. D.*

In the above Example, we have, by this Rule, the numerical Operation thus:

Multiply	845	From	1000000
By	700	Subtract	591500
	591500		408500
Multiply	591500	4085)396305(97 nearly, = GD.	
By	67	28655	
	4140500	60	
	3549000		
	39630500		

Now, by Prop. 1, Caſe 5, As 845 : 1000 :: 97 Feet : 114.8 Feet, nearly, = DE, to which adding the Height of the Eye, 5 Feet, we have the whole Height CE = 119.8 Feet.

112. *Caſe 3.* If the Numbers, cut by the Plumb-Line, are both on the Siniſtra-Side, then the Rule is, multiply * the leaſt Number cut on the Quadrat by FG, and divide the Product by the Difference of the

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* *Demonſtration.* Let a = each Side of the Quadrat, l = the leſſer, g = the greater Number cut on the Siniſtra-Side of the Quadrat, m = FG, x = GD; then FD $= m+x$. By Prop. 1, Caſe 5, As $a : l :: m+x : \frac{lm+lx}{a}$ = DE: Again, $a : g :: x : \frac{gx}{a}$ = DE. $\therefore \frac{gx}{a} = \frac{lm+lx}{a}$. Hence $gx = lm+lx$. Therefore, $x = \frac{lm}{g-l}$. *Q. E. D.*

two Numbers found on the Quadrat; the Quotient will be equal to GD. Then find DE by Propoſition 1, Caſe 5.

Example. Let the Numbers, cut on the Siniſtra-Side of the Quadrat, be 49 and 74, counting the Side of the Quadrat 100, and FG = 78 Feet.

Solution. Proceeding by the above Rule.

Multiply	78	From	74	25)3822(152.88
By	49	Subtract	49	
	702		25	132
	312			72
	3822			220
				200

Now, by Prop. 1, Caſe 5, As 100 : 74 :: 152.88 Feet : 113.1 Feet = DE; to which adding the Height of the Eye, it gives CE. Or we may ſay, As 100 : 49 :: 78+152.88 : DE, as above.

113. *Caſe* 4. But if the Numbers, cut by the Plumb-Line, are both on the Dextra-Side, the Rule* is the ſame as in the laſt Caſe.

Example. Let FG = 54, the Numbers cut on the Dextra-Side of the Quadrat 82 and 31, counting the Side 100.

Numerical

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* *Demonſtration.* Let a = each Side of the Quadrat, l = the leſſer g = the greater Number cut on the Dextra-Side of the Quadrat, m = FG, x = GD; then $m+x$ = FD. By Prop. 1, Caſe 5, As $l : a :: x : \frac{ax}{l}$ = DE; and, As $g : a :: m+x : \frac{am+ax}{g}$ = DE. Hence $\frac{x}{l} = \frac{m+x}{g}$. $\therefore x = \frac{lm}{g-l}$ = GD. *Q. E. D.*

Numerical Operation.

Multiply	54	From	82	51)1674(32.82 =
By	31	Subtract	31	GD.
				144
	54	Diviſor	51	
	162			420
Dividend	1674			120
				18

By Caſe 5, As 31 : 100 :: 32.82 Feet : 106 Feet, nearly, = DE. Or, As 82 : 100 :: 54 + 32.82 : DE, as before.

114. *Propoſition* 3. To find the Height of an acceſſible Object, AD, when the Ground riſes from it.

Pl. III. Fig. 13. *Method.* Let F be the Place of the Eye of a Perſon ſtanding at any convenient Place, G, on the Hill; then, ſuppoſing FC an horizontal Line, by a Quadrant, take the ∠ of Altitude of the Top at D, which will be equal to ∠ CFD, which, for Example-Sake, we will ſuppoſe = 20 d. and at the ſame Place take the ∠ of Depreſſion = 40 d. = BFC, (of a Mark B, ſet up from A on the Side of the Object AD, = the Height of the Eye FG;) then meaſure from G to A, ſuppoſe = 100 Feet: This done, the

Conſtruction is, Draw a Line AH, and erect a ⊥ AE. Take AB = Height of the Eye, ſuppoſe 5 Feet; make ∠ FBE = 90 d. — 40 d. and BF = 100 Feet: Laſtly, make ∠ BFD = 40 d. + 20 d. = 60 d. then is AD the Height of the Object required.

Calculation. Firſt 90 d. — ∠ DFC 20 d. = ∠ FDC 70 d. Hence, by Trigonometry, As S. ∠ FDB 70 d. : BF 100 Feet :: S. ∠ BFD = 60 d. : BD = 92.16 Feet; to which adding AB, the Height of the Eye, 5 Feet, gives AD, the Height of the Object required, = 97.16 Feet.

115. *Propoſition* 4. To find the Height of an acceſſible Object on the Top of a riſing Ground, as BD.

Suppoſe, wanting to find the Height of the Object BD, ſtanding at A, the Eye at E, 5 Feet high, I take the ∠ of Elevation FED = 50 d. and (BC being ſet off = AE = 5 Feet) the ∠ of Elevation FEC = 10 d. then, meaſuring from A to B, I found AB = EC = 60 Feet. Pl. III. Fig. 14.

Conſtruction. Make ∠ FEC = 10 d. and EC = 60 Feet: Through C let fall the ⊥ GF, and produce FC upwards at Pleaſure; make ∠ FED = 50 d. then is BD manifeſtly the Height of the Object required.

Calculation. Firſt, 90 d. — ∠ FED 50 d. = 40 d. = ∠ D; and ∠ FED 50 d. — ∠ FEC 10 d. = 40 d. = ∠ CED. By Trigonometry, As S. ∠ D 40 d. : EC 60 Feet :: S. ∠ CED 40 d. : CD = 60 Feet; to which adding the Height of the Eye BC 5 Feet, it gives the Object's Height BD = 65 Feet.

116. *Propoſition* 5. To find the Height of an Object on the Top of a Hill which is inacceſſible.

Suppoſe, wanting to find the Height of an Object, DE, on a Hill, CD, which is of too ſteep an Aſcent to be conveniently mounted, I therefore ſtood at a convenient Place on the horizontal Plane, *viz.* the Eye being at B, I took Angles of Altitude CBE = 61 d. CBD = 39 d. then, meaſuring backwards, took the Diſtance BA = 50 Feet; and, the Eye being at A, the Angles of Altitude CAE = 42 d. and CAD = 22 d. Hence it is required to find the Height of the Object DE. Pl. III. Fig. 15.

N. B. The Height of the Eye is of no Conſequence in this Propoſition.

Conſtruction. Draw a horizontal Line ABC, and make ∠ CAD = 22 d. CAE = 42 d. ſet off AB = 50 Feet, and make the ∠ CBE = 61 d. ∠ CBD = 39 d. join the Points of interſection DE, then is DE the Height of the Object required.

Solution. The ∠ ABE = 180 d. — ∠ CBE 61 d. = 119 d. and ∠ BAE 42 d. + ∠ ABE 119 d. = 161 d. therefore ∠ AEB = 180 d. — 161 d. = 19 d.

Or ſhorter thus: Producing ED to C, the △ AEC is right-∠d by Suppoſition; and therefore the ∠s AEC and BEC are Complements of the ∠s CAE, CBE; ∵ the ∠ AEB being the Difference of theſe Complements muſt be alſo = Difference of theſe ∠s; ∵ ∠ CBE 61 d. — ∠ CAE 42 d. = ∠ AEB = 19 d. as before.

Hence, in △ ABE, As S. ∠ AEB 19 d. : AB 50 Feet :: S. ∠ BAE 42 d. : BE = 102.8 Feet.

Now the ∠ CBE 61 d. — ∠ CBD 39 d. = ∠ DBE = 22 d. and, the △ BCD being right-angled at C, the ∠ BDE = 90 d. — ∠ CBD 39 d. = 51 d. and ∠ BDE = 180 d. — ∠ BDC 51 d. = 129 d. Hence, As S. ∠ BDE 129 d. or Sine of its Supplement 51 d. : BE 102.8 Feet :: S. ∠ DBE 22 d. : DE = 49.5 Feet, which was to be found.

117. *Propoſition* 6. To find the Height of an inacceſſible Object, when we can only meaſure on the Side of a Hill on which it ſtands.

Pl. V. Fig. 1. EXAMPLE. Wanting to find the Height of an inacceſſible Object, CD, at a convenient Place, B, I ſet up a Pole or Mark, BE, equal to the Height of my Eye, AF, ſuppoſe 5 Feet. Conceiving FG and EH ∥ to the Horizon, ſtanding firſt at A, the Eye at F, I took the ∠ of Altitude GFD = 46° 30′ and GFE = 33 d. then meaſuring from A to B (or F to E) I found that Diſtance = 38 Feet; then, with the Eye at E, I took the ∠ HED = 61° 30′, and the ∠ to the Foot of the Object, *viz.* HEC, = 18 d. Hence it is required to find the Height CD.

Conſtruction. Draw a Line, FG, to repreſent a horizontal one; make ∠ GFE = 33 d. GFD = 46 d. 30′; ſet off FE = 38 Feet; draw EH ∥ FG; then make ∠ HEC = 18 d. and HED = 61 d. 30′; from the Point of Interſection, D, of the Lines FD, ED, let fall the ⊥ DC, and it is evidently the Height of the required Object above the Hill.

Calculation. The ∠ GFD 46 d. 30′ — ∠ GFE 33 d. = 13 d. 30′ = ∠ EFD; and, for the Reaſon ſhewn in the laſt Propoſition, the ∠ HED 61 d. 30′ — ∠ GFD 46 d. 30′ = ∠ FDE = 15 d. Hence, in △ FDE, we have, As S. ∠ FDE 15 d. : FE 38 Feet :: S. ∠ EFD 13 d. 30′ : ED = 34.27 Feet. Now, ∠ HED 61 d. 30′ — ∠ HEC 18 d. = ∠ CED = 43 d. 30′; and the △ EHC being right-angled at H, the ∠ ECH = 90 d. — ∠ HEC 18 d. = 72 d. But ∠ ECH is the Supplement of ∠ ECD.

Hence, by Trig. As S. ∠ ECD, or, which is equal, the Sine of its Supplement, ECH 72 d. : ED :: S. ∠ CED 43 d. 30′ : CD 24.88 Feet. *Q. E. I.*

118. *Propoſition* 7. To find the Height of an inacceſſible Object at the Foot of a Hill, when we can only meaſure down the Hill towards it.

Pl. V. Fig. 2.

Example. Wanting to find the Height of the Object CF, the Eye being at A, I took the ∠ of Altitude EAF = 18½ d. and the ∠ of Depreſſion EAB = 21½ d. then, meaſuring towards the Foot of the Object C, the Diſtance AB = 38 Feet; my Eye being then at B, I took the ∠ of Altitude DBF = 48 d. and of Depreſſion DBC = 31 d. Quære the Height of the Object FC.

Conſtruction. Draw a horizontal Line AE, make ∠ EAF = 18½ d. and ∠ EAB = 21½ d. then ſet off AB = 38 Feet: Through B draw BD ∥ AE, make ∠ DBF = 48 d. and ∠ DBC = 31 d. From the Point of Interſection, F, let fall the ⊥ FE, which produce to C, to interſect the Line BC, then is CF the Height of the required Object.

Calculation. The ∠ EAF 18½ d. + ∠ EAB 21½ d. = ∠ FAB = 40 d. and, for the Reaſon ſhewn in Prop. 5, the ∠ DBF 48 d. — ∠ EAF 18½ d. = ∠ AFB = 29½ d. Now, by Trigonometry, As S. ∠ AFB 29½ d. : AB 38 Feet :: S. ∠ FAB 40 d. : BF = 49.61 Feet.

The ∠FBD 48 d. + ∠DBC 31 d. = ∠FBC 79 d and 90 d.—∠DBC 31 d. = ∠BCD = 59 d. Hence, by Trigonometry, As S. ∠ BCD 59 d. : BF 49.61 Feet :: S.∠CBF 79 d. : FC = 56.8 Feet. *Q. E. I.*

119. *Propoſition* 8. To find the Height of an inacceſſible Object, CD, ſtanding on a horizontal Plane, BC, when I can only meaſure on the Side of an oppoſite Hill, BA.

Pl. V. Fig. 3. EXAMPLE. Standing at A, I took the ∠s of Depreſſion EAD = 7 d. EAC = 19 d. EAB = 31 d. and, by meaſuring, found AB = 87 Feet. Quære the Height of the Object CD, and BC the Diſtance from the Foot of the Hill.

Conſtruction. Draw a horizontal Line AE, make ∠EAD = 7 d. ∠EAC = 19 d. and ∠ EAB = 31 d. ſet off AB = 87 Feet, and draw BC ∥ AE; at the Interſection C raiſe CD ⊥ to BC; then is CD the Height required.

Calculation. The ∠BCA = ∠EAC = 19 d. and ∠FBA being = EAB = 31 d. and the ∠ FBA being the Supplement of ∠ABC, we have, by Trigonometry, As S.∠BCA 19 d. : AB 87 Feet :: S. of ∠ ABC or Sine of the ſupplemental ∠ FBA 31 d. : AC = 137.6 Feet.

The ∠ EAC 19 d.— ∠ EAD 7 d. = ∠DAC = 12 d. and 90 d. — ∠ EAD 7 d. = ADE = 83 d. Hence, As S. ∠ADC or Sine of its Supplement 83 d. : AC 137.6 Feet :: S.∠DAC 12 d. : DC = 28.83 Feet. *Q. E. I.*

If the Diſtance BC be required, then ∠EAB 31 d. — ∠EAC 19 d. = ∠BAC = 12 d. Hence, As S.∠ BCA 19 d. : AB 87 Feet :: S. ∠ BAC 12 d. : BC 55.56 Feet. *Q. E. I.*

120. *Propoſition* 9. To find the Height of an Object oppoſite a Houſe or Tower, by Angles taken at the Windows of the Houſe, or at Top and Bottom of the Tower.

EXAMPLE. From a Window at G, my Eye being 5 Feet above the horizontal Plane AB, I took the ∠

of Altitude CGE = 41d. 10′; then, going up to another Window, F, 15 Feet higher, I took the $\angle$ of Altitude DFE = $27\frac{1}{2}$d. Quære the Height of the Object BE.

Conſtruction. Draw a horizontal Line AB, and erect the $\perp$ AF = 15+5 = 20 Feet; make AG = 5 Feet; draw FD and GC $\parallel$ to AB; make the $\angle$ CGE = 41d. 10′, and $\angle$ DFE = 27 d. 30′; then is the Point of Interſection E the Top of the Object: $\because$ let fall the $\perp$ EB, and it is manifeſtly the Height required. Pl. V. Fig. 4.

Calculation. The $\angle$ CGE 41d. 10′ — $\angle$ DFE 27d. 30′ = 13 d. 40′ = $\angle$ FEG: and $\angle$ EFD 27 d. 30′ + 90 d. = 117 d. 30′ = $\angle$ EFG. Hence we have, As S. $\angle$ FEG 13d. 40′ : GF 15 Feet :: S. $\angle$ GFE 117 d. 30′ : GE 56.32 Feet. And, As Sine of 90 d. : GE 56.32 Feet :: S. $\angle$ CGE 41d. 10′ : CE 37.07 Feet. Hence BE = 37.07+5 = 42.07 Feet, the Height which was required.

If the Diſtance AB (GC) be required, As Radius 90 d. : GE 56.32 Feet :: Coſine $\angle$ CGE 41d. 10′ (or S. $\angle$ GEC 48 d. 50′) : GC = AB = 42.39 Feet.

121. *Propoſition* 10. To find the Height of an Object by the Barometer.

We think we cannot oblige our Readers more than by giving them the following Paper (from the *Philoſ. Tranſ.* Vol 64, Part 1, 1774) as a Solution to this Propoſition, *viz.*

M. De Luc's *Rule, for meaſuring Heights by the Barometer, reduced to the* Engliſh *Meaſure of Length, and adapted to* Fahrenheit's *Thermometer and other Scales of Heat, and reduced to a more convenient Expreſſion. By the Aſtronomer-Royal.*

M. *De Luc, F. R. S.* in a large and valuable Treatiſe upon the Barometer and Thermometer, lately publiſhed at Geneva, the Reſult of many Years Labour and Study, has given a Rule for the Meaſurement of Heights by the Barometer, deduced from his Experiments, and far more accurate than any publiſhed before; ſince it appears that he could determine Heights by it generally to 10 or 15 Feet, and that the Error ſeldom, if ever, amounted to double that Quantity. This valuable Degree of Exactneſs he has obtained principally by detecting the Faults of the common Barometer, and, in Conſequence, improving the Conſtruction of it, and by introducing the Uſe of the mercurial Thermometer, to accompany that of the Barometer. The principal Faults, which he found in the common Barometers, aroſe from the Repulſion of the Quickſilver by the Glaſs Tube, from Air and Moiſture admitted into the Tube, and from the Variations of the Denſity of Quickſilver by Heat and Cold; another very conſiderable Error aroſe, in calculating Heights from the Barometer, by not allowing for the Changes of the Denſity of the Air, whoſe Gravity affords us this Meaſure of Heights, owing to Heat and Cold. The firſt Cauſe of Error (that of the Repulſion of the Tubes) he remedied by ſubſtituting a Syphon-Barometer inſtead of the ſimple upright Tube, the Repulſion of the two Legs of the Syphon counteracting itſelf: The Error, ariſing from Air and Moiſture in the Tube, he cured by boiling the Quickſilver after it was put into the Tube, and other Precautions: The Errors, in the Eſtimation of the Heights, ariſing from the Changes of the Denſity of the Quickſilver and Denſity of the Air, by Heat and Cold, he ſhews how to correct by Allowances, depending on two Thermometers, one attached to the Frame of the Barometer itſelf, and the other made to be expoſed to the open Air, to ſhew its Degree of Heat; which Thermometers are to be noted both at the Top and Bottom of the Hill. Laſtly, by a great Number of Experiments, made with accurate Barometers and Thermometers of his own Conſtruction, he has deduced a Rule for calculating the Heights of Places, the Exactneſs of which he has ſufficiently proved by

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a large Table of Experiments. But this Rule is expreſſed in *French* Meaſure, and is adapted either to a Thermometer, whoſe freezing Point is 0, and that of boiling Water 80, or to Thermometers of particular Scales. It may be therefore uſeful to reduce M. *De Luc*'s Rule to Engliſh Meaſure, and to adapt it to the Thermometer of *Fahrenheit*'s Scale, which is generally uſed in this Country.

122.* M. *De Luc*, in the Winter-Seaſon, heated the Air of his Room to as great a Degree as he could, and noted the Riſe of the Barometer, owing to the Diminution of its Denſity, or ſpecific Gravity, by Heat; he alſo noted the Height of the Thermometer, both before and after the Room was heated. Hence he deduced a Rule, that, when the Barometer is at 27 *French* Inches, which was the Caſe in this Experiment, an Increaſe of Heat, from freezing to that of boiling Water, will raiſe the Barometer 6 Lines, or $\frac{1}{54}$th Part of the Whole. It is eaſy to ſee, that, when the Barometer is higher than 27 *French* Inches, this Variation will increaſe in the ſame Proportion, or will be always $\frac{1}{54}$th of the Height of the Barometer; therefore, if the Height of the Barometer be called B, the Riſe of the Barometer, for an Increaſe of Heat, from freezing to boiling Water, will be $\frac{B}{54}$; and, as it will be leſs for a leſs Difference of Heat, therefore, if the Number of Degrees, marked on the Thermometer, between freezing and boiling Water, be called K, and the Riſe of the Thermometer, from any given Point, be called H, the correſpondent Riſe of the Barometer will be $\frac{B}{54}\times\frac{H}{K}$, by the Increaſe of Heat from the given Point, by the Number of Degrees H. If the Heat, inſtead of increaſing, was to decreaſe, then H would ſignify ſo many Degrees Decreaſe of Heat, and the Barometer would ſink by $\frac{B}{54}\times\frac{H}{K}$. The fixed Temperature of Heat, to which M. *De Luc* thought beſt to reduce his Obſervations of the Barometer, is $\frac{1}{8}$th of the Interval, from freezing to boiling Water, above the former Point; and, if the Thermometer was higher than this Degree, he ſubtracted

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* For the Sake of ſuch of our Readers as do not underſtand algebraic Expreſſions, we have printed the greater Part of this Eſſay of Mr. MASKELYNE's on a leſs Type, that they may not be deterred from reading the practical Part, which is printed on a larger Type, and may be eaſily underſtood.

$\frac{B}{54} \times \frac{H}{K}$; if it was lower, he added it to the obſerved Height of the Barometer; and thus he obtained the exact Height of the Barometer, ſuch as it would have been if the Denſity of its Quickſilver had been the ſame as anſwers to the fixed Degree of Temperature. He thus corrected the Height of both his Barometers, (that at the Bottom and that at the Top of the Hill,) for the particular Degree of Heat, indicated by a Thermometer attached to the Barometer, at each Station; for it might and would commonly happen, that the Degree of Heat would be different at the two Stations. The Heights of the Barometers, thus corrected, were what he made Uſe of in his ſubſequent Calculations. Calling theſe two Altitudes of the Barometer B and b, putting Log. B and Log. b for the Logarithms of B and b, taking only the four firſt Places of Figures, after the Characteriſtic, or conſidering the remaining Figures as Decimals, and putting C for the mean Height of a Thermometer, expoſed to the Air at Top and Bottom of the Hill, the freezing Point being 0, and the Point of boiling Water at 80, he finds, by his Experiments, that the Height of the Hill will be given in *French* Toiſes, when C is 16¾, by ſimply taking the Difference of the Logarithms of the Heights of the Barometer, or will be equal Log. B — Log. b; and in any other Degree of Heat will be greater or leſs, in Proportion as the Rarity of the Air is greater or leſs than the fixed Temperature; or greater or leſs by $\frac{1}{215}$th Part of the Whole, for every Degree of the Thermometer reckoned from the fixed Temperature 16¾; and conſequently the Height of the Place will be expreſſed generally, in French Toiſes, by this *Formula*, $\text{Log. B} - \text{Log } b + \overline{\text{Log.B} - \text{Log.}b} \times \frac{C - 16\frac{3}{4}}{215} = \overline{\text{Log. B} - \text{Log.}b} \times \overline{1 + \frac{C - 16\frac{3}{4}}{215}}$. To reduce this *Formula* to Engliſh Meaſure, and to the Scale of *Fahrenheit's* Thermometer, we ſhould firſt premiſe ſome Particulars. The *French* Foot is to the *Engliſh* Foot as 1.06575 to 1, as was found by a very accurate Experiment: See *Phil. Tranſ.* Vol. LVIII. for 1768, P. 326: And it is well known, that the Point of freezing, on *Fahrenheit's* Thermometer, is at 32, and that of boiling Water at 212, or the Interval between them 180 Degrees. But M. *De Luc's* Point of boiling Water, 80, was marked when the Barometer was at 27 *French* Inches; and it is the Cuſtom of our principal *Engliſh* Workmen to mark the Point of boiling Water, 212, on *Fahrenheit's* Thermometer, when the Barometer ſtands at 30 Inches, which is equal to 28 Inches 1.8 Lines *French* Meaſure; or 13.8 Lines higher than M. *De Luc's* Barometer, when he ſet off the Point of boiling Water on his Thermometers; and it is well known, that the Heat of boiling Water varies with the Weight of the Atmoſphere. M. *De Luc* finds, by his Experiments, this Rule, that an Increaſe of 1 Line, in the Height of the Barometer, raiſes the Quickſilver of the Thermometer, placed in boiling Water, by $\frac{1}{1134}$th Part of the Interval between the freezing Point and that of boiling Water: He afterwards, indeed, found that this Rule would not anſwer for ſuch large Variations of the Barometer as take Place in aſcending to very great Heights above the Earth's Surface; (*vid. Eſſai ſur les variations du chaleur de l'eau bouillie*;) but it is accurate enough for any ſmall Variation of the Barometer, on one Side or other of its mean Height, in theſe loweſt Regions of the Atmoſphere. The Change, therefore, of the Boiling-Point, on *Fahrenheit's* Scale, for a Change of 1 Line in the Barometer, will be $\frac{180}{1134} = 0.16$; therefore 13.8 Lines will cauſe $0.16 \times 13.8 = 2.2$ Degrees of *Fahrenheit's* Scale; and a Thermometer, whoſe Point of boiling Water was marked 212, when the Barometer ſtood at 30 *Engliſh* Inches, (= 28 Inches 1.8 Line *French* Meaſure,) will, when the Barometer deſcends to 27 *French* Inches, ſink 2.2 Degrees in boiling Water, or to 209.8, or, in round Numbers, to 210 Degrees, which is diſtant only 178 from 32, the Point of freezing. Hence, an Extent of 80 d. of M. *De Luc*'s Thermometer, anſwers to an Extent of 178 of our *Fahrenheit's* Thermometer: and, putting F for the Degrees of this Thermometer, (correſponding to C of M. *De Luc*'s,) we ſhall have, $C : F - 32 :: 80 : 178$, and $C = \overline{F - 32} \times \frac{80}{178}$, which, ſubſtituted in M. *De Luc's Formula*, gives $\overline{\text{Log. B} - \text{Log. } b} \times \overline{1 + \frac{C - 16\frac{3}{4}}{215}}$

$$= \overline{\text{Log. B} - \text{Log. } b} \times \overline{1 + \frac{\overline{F - 32} \times \frac{80}{178} - 16\frac{3}{4}}{215}} = \overline{\text{Log. B} - \text{Log. } b} \times \overline{1 + \frac{80}{178 \times 215}\,\overline{F - 32 - \frac{178}{80}\,16.75}} = \overline{\text{Log. B} - \text{Log. } b} \times \overline{1 + \frac{F - 32 - 37.27}{478.38}} = \overline{\text{Log. B} - \text{Log. } b} \times \overline{1 + \frac{F - 69.27}{478.38}}.$$

Where the Anſwer will ſtill come out in *French* Toiſes, though adapted to *Fahrenheit*'s Thermometer. To bring it out in *Engliſh* Fathoms; (or Meaſure of 6 Feet,) multiply the above Expreſſion by 1.06575, and we ſhall have $\overline{\text{Logarithm B} - \text{Logarithm } b} \times \overline{1 + \frac{F - 69.27}{478.38}} \times 1.06575 = \overline{\text{Log. B} - \text{Log. } b} \times \overline{409.11 + F} \times \frac{1.06575}{478.38} = \overline{\text{Log. B} - \text{Log. } b} \times \frac{409.11 + F}{448.87}$, or, in round

Numbers, $= \overline{\text{Log. B} - \text{Log. } b} \times \overline{\frac{409 + \text{F}}{449}} = \overline{\text{Log. B} - \text{Log. } b} \times \overline{1 + \frac{\text{F} - 40}{449}}$; which will expreſs the Height between the two Stations in *Engliſh* Fathoms.

123. In the foregoing Expreſſions, B and b, as has been mentioned before, ſignify Heights of the Barometer at the lower and higher Stations, both corrected according to M. *De Luc's* Directions, for the Difference of Heat between a fixed Temperature (namely, ⅛th of the Interval between freezing and boiling Water) and the preſent Heat, indicated by the Thermometer attached to the Barometer at each Station; but it is not neceſſary to correct *both* Barometers for the Effect of Heat, but only one, for the Difference of Heat of the two; which will be more convenient, alſo, on another Account, becauſe the Difference of Heat, at the two Stations, will be generally ſmall, and the Correction, to reduce one Barometer to the Heat of the other, will conſequently be ſmall alſo; whereas, the Difference of the preſent Heat and the fixed Temperature, and conſequently the Correction of both Barometers, may be frequently very conſiderable. This is evident: Becauſe, if the Heat of the Barometers, at both Stations, was the ſame, (however different from the fixed Temperature choſen by M. *De Luc*,) no Correction would be neceſſary; the Mercury in the Barometer, in both Stations, being expanded in the ſame Proportion, and conſequently the Difference of the Logarithms of its Height, at both Stations, being the ſame as if the Heat of both Barometers had agreed with that of the fixed Temperature. I ſhall now, therefore, ſuppoſe the upper Barometer is to be corrected, to reduce it to the Temperature of the lower one, and that b ſignifies the Height of this Barometer, as obſerved, and not yet corrected: The Correction, from what has been ſaid above, calling D the Difference of Height of the Thermometer attached to the Barometer at the two Stations, will be $\pm \frac{\text{D}\,b}{54\,\text{K}}$, according as the Thermometer ſtands higheſt at the lower or upper Station; and the upper Barometer, corrected, inſtead of b, will be $b \pm \frac{\text{D}\,b}{54\,\text{K}}$, which, ſubſtituted in the *Formula*, gives $\overline{\text{Log. B} - \text{Log.} \left(b \pm \frac{\text{D}\,b}{54\,\text{K}}\right)} \times \overline{1 + \frac{\text{F} - 40}{449}}$. But the Correction, on Account of the Difference of Heat of the Barometer at the two Stations, may be re-

duced to a ſtill eaſier Expreſſion, in which the variable Quantity, *b*, the Height of the upper Barometer, ſhall not appear. The Fluxion of a Logarithm is, to the Fluxion of its natural Number, as the Modulus of the Syſtem to the natural Number; and 4343 is the Modulus of the common Logarithms, when the four Places, next following the Characteriſtic, are taken as whole Numbers inſtead of Decimals, which is meant to be done in the Uſe of the foregoing *Formula*. Therefore $\frac{D\,b}{54\,K}$ being very ſmall, with Reſpect to *b*, we ſhall have, Variation of Log. b : Variation of b $\left(=\frac{D\,b}{54\,K}\right)$ :: 4343 : b, very nearly; and thence Variation of Log. $b = \pm\frac{D\,b}{54\,K}\times\frac{4343}{b} = \pm\frac{4343\,D}{54\,K}$. Which (putting $K = 178$) $= \pm\,0.452\,D$. Hence Log. $\left(b \pm \frac{D\,b}{54\,K}\right) = \text{Log. } b \pm 0.452\,D$; which, being ſubſtitited in the *Formula* above, will give the Difference of Height of the two Stations in *Engliſh* Fathoms, in a more convenient Expreſſion; namely, $\overline{\text{Log. } B - \text{Log. } b \mp 0.452\,D} \times \overline{1 + \frac{F-40}{449}}$; where the upper Sign, −, is to be uſed when the Thermometer of the Barometer is higheſt at the lower Station, and the lower Sign, +, is to be uſed when the ſaid Thermometer is loweſt at the lower Station. The firſt Caſe will be moſt common; eſpecially where the Difference of Height of the two Stations is conſiderable. It ſhould alſo be obſerved, that when F, the Height of *Fahrenheit's* Thermometer, is leſs than 40 d. $+\frac{F-40}{449}$ becoming negative or ſubtractive, it muſt be applied in the Calculation accordingly.

124. It may perhaps be convenient to repeat here the Meaning of the algebraic Terms uſed in the foregoing *Formula*, that any Perſon may make Uſe of it, without having Occaſion to recur to the foregoing Inveſtigation. B ſignifies the obſerved Altitude of the Barometer at the lower Station, and *b* that at the upper Station: Log. B and Log. *b* ſignify their Logarithms, taken out of the common Tables, by aſſuming the four firſt Figures next following the Characteriſtic as whole Numbers, and conſidering the three remaining Figures, to the Right-Hand, as Decimals: D ſignifies the Difference of Height of *Farenheit's* Thermometer, attached to the Barometer at the Top and Bottom of the Hill: and F ſignifies the Mean of the two Heights of

Fahrenheit's Thermometer, expoſed freely for a few Minutes to the open Air in the Shade, at the Top and Bottom of the Hill.

125. The *Formula* for the Meaſure of Heights may alſo be changed, and adapted to Thermometers of particular Scales, for the Convenience of Calculation, as M. *De Luc* has done; but theſe Scales will be different from his. The Thermometer, attached to the Barometer, had better be divided with the Interval between freezing and boiling Water, conſiſting of 81.4 d. ($= 180 \times .452$); the Freezing-Point may be marked 0, and the Point of boiling Water will be 81.4; for, then, if the Difference of Height of this Thermometer, at the two Stations, be called *d*, we ſhall have $d = 0.452 \times D$; for $d : D :: 81.4 : 180 :: 0.452 : 1$, and the Number of Degrees, expreſſed by *d*, will ſhew immediately the Correction for the Difference of Heat of the two Barometers. If the Thermometer, deſigned to ſhew the Temperature of the Air, be divided with the Interval between freezing and boiling Water $= 200$, and the Freezing-Point be marked -9, and the Boiling-Point $+191$, and the Heights of this Thermometer, at the two Stations, be called G and I, we ſhall have $\frac{F-40}{449} = \frac{G+I}{2 \times 500} = \frac{G+I}{1000}$. For $F-40 = F-32-8$ is the Height of *Fahrenheit's* Thermometer, reckoned from 8 Degrees above freezing, and $449 : 500 :: 180 : 200 :: 8 : 9$; and the Fraction $\frac{F-32-8}{449}$, if both the Numerator and Denominator be increaſed in the Ratio of 449 to 500, will become $= \frac{\overline{F-32-8} \times \frac{500}{449}}{500} = \frac{\overline{F-32}\,\frac{500}{449} - 9}{500} = \frac{G+I}{2 \times 500} = \frac{G+I}{1000}$, becauſe $\frac{G+I}{2} + 9 = \overline{F-32} \times \frac{500}{449}$. Therefore, if the Thermometer of the Barometer has the freezing-Point marked 0, and the Point of boiling Water 81.4, and the Difference of its Height, at the two Stations, be called *d*; and if the Thermometer, for meaſuring the Temperature of the Air, be divided with the Interval of 200, between the Freezing-Point and that of boiling Water, and the firſt be marked -9 and the latter $+191$, and the Degrees, ſhewn by this at the two Stations, be called G and I; the *Formula*, that will give the Height of the upper Station above the lower one, in *Engliſh* Fathoms, will be $\overline{\text{Log.}\,B - \text{Log.}\,b \mp d} \times \overline{1 + \frac{G+I}{1000}}$; which conſequently, multiplied by 6, will give the Height in *Engliſh* Feet. It is to be obſerved, as before, that $-d$ or $+d$ is to be uſed, according as the Thermometer, attached to the Barometer,

is higheſt at the lower or upper Station; and, if G and I ſhould happen to fall below o of the Scale, or to be ſubtractive, they muſt be applied accordingly in the Calculation.

126. I ſhall now add Nothing more, but give the Rule for finding Heights by the Barometer, according to the *Formulæ* delivered above, in common Language; firſt, as adapted to *Fahrenheit*'s Thermometer, and, next, as adapted to the two Thermometers of particular Scales. Take the Difference of the tabular Logarithms of the obſerved Heights of the Barometer, at the two Stations, conſidering the 4 firſt Figures, excluſive of the Index, as whole Numbers, and the 3 remaining Figures, to the Right, as Decimals, and ſubtract or add $\frac{452}{1000}$th of the Difference of the Altitude of *Fahrenheit*'s Thermometer, attached to the Barometer at the two Stations, according as it was higheſt at the lower or upper Station; thus you will have the Height of the upper Station above the lower, in *Engliſh* Fathoms, nearly; which is to be corrected as follows: Make this Proportion: As 449 is to the Difference of the mean Altitude of *Fahrenheit*'s Thermometer, (expoſed to the Air at the two Stations, from 40°,) ſo is the Height of the upper Station, found nearly, to the Correction of the ſame; which added or ſubtracted, according as the mean Altitude of *Fahrenheit*'s Thermometer was higher or lower than 40°, will give the true Height of the upper Station above the lower, in *Engliſh* Fathoms; and, multiplied by 6, will give it in *Engliſh* Feet.

127. The ſame Rule, adapted to the Thermometers of particular Scales, is this:

Take the Difference of the tabular Logarithms of the obſerved Heights of the Barometer, at the two Stations, conſidering the 4 firſt Figures (excluſive of the Index) as whole Numbers, and the 3 remaining Figures, to the Right, as Decimals; and ſubtract or add the Difference of a Thermometer of a particular Scale, attached to the Barometer at the two

Stations, according as it was higheſt at the lower or upper Station, and you will have the Height of the upper Station above the lower one, in *Engliſh* Fathoms, nearly; which is to be corrected as follows: Make this Proportion: As 1000 is to the Sum of the Altitudes of a Thermometer of a particular Scale, expoſed to the Air at both Stations, ſo is the Height of the upper Station above the lower, found nearly, to the Correction of the ſame; which added or ſubtracted, according as the Sum of the Altitudes of the Thermometer, expoſed to the Air, is poſitive or negative, will give the true Height of the upper Station above the lower, in *Engliſh* Fathoms; and, multiplied by 6, will give it in *Engliſh* Feet.

NEVIL MASKELYNE.

128. As we could not abridge this Paper without injuring it, we doubt not but that the ingenious Author would rather ſee it printed here entire. *Samuel Horſley, L. L. D.* has alſo, in the ſame Tranſactions, a very curious Paper on this Subject, on M. *De Luc*'s Principles; but Room will only permit us to give ſome of the Tables he has calculated, (to render the Practice eaſier,) and to illuſtrate their Uſe in a different Manner, which, perhaps, may be more eaſily underſtood by the practical Surveyor than that of the learned Author.

129. The Heights of Hills may be eaſily computed, by Means of the following Tables, by theſe Rules: Firſt, take out the Logarithms of the Heights of the Barometer, at Top and Bottom of the Hill, from a Table which has them to 7 decimal Places; but the Index and all the decimal Places are here to be conſidered as whole Numbers. Subtract the leſſer Logarithm from the greater, and divide the Remainder by 1000, which is readily done, by cutting off 3 Figures on the Right-Hand; the Quotient will give a Number of *Engliſh* Fathoms, (each 6 Feet,) which we will call the Groſs-Height.

Example. Let the lower Barometer be ſuppoſed to ſtand at 29.68 Inches, the upper at 25.28 Inches:

The Log. of 29.68 =	14724639	taken as whole
Log. of 25.28 =	14027771	Numbers.
Their Diff. ÷ by 1000 =	696.868 =	Groſs-Height of the Hill, in Fathoms.

N. B. What is here called the *Groſs-Height* of the Hill is the *true Height*, when the Thermometer ſtands at 40° both at Top and Bottom of the Hill; but, if not, it will require two Corrections, as will be explained in the following Rules.

130. Secondly, With the Difference of the Temperatures of the Quickſilver, (as ſhewn by the Thermometers in the Caſes of your portable Barometers,) take out of Table I. from the Column of Fathoms, the Equation correſponding thereto, which call the *Equation of the Mercury*: But, if the Thermometer, in the *higher Station*, gives a *warmer Air* than in the *lower Station*, then we muſt *add* this *Equation to*, otherwiſe *ſubtract* it *from*, the *Groſs-Height*, and the Sum, or Remainder, will be the *Approximate-Height*.

Example. Let the Groſs-Height of a Hill, found by the laſt Article, be 696.868, and let us ſuppoſe the Thermometer, at the Bottom of the Hill, ſtood at +57, and, at the Top of the Hill, at +43, their Difference is 14.

Againſt 10, in the 3d Column of Table I. is	4.518
—— 4 —— —— ——	1.807
The Sum is the Equation of the Mercury	6.325

From the Groſs-Height of the Hill	696.868
Subtract the Equation of Mercury	6.325
Gives the Approximate-Height	690.543

131.

131. Thirdly, Add together the Temperatures of the Air, ſhewn by the Thermometers (in open Air) under their proper Signs, and call Half the Sum the *Temperature of the Air*, being the mean State of the Air. With this Temperature, from Table II. find Part of the Equation for the Air thus: Look at Top for the Number of Decades (or Tens) next leſs than the Temperature of the Air; and from that Column collect (by repeated Entries, if neceſſary) the Corrections for the Units, Tens, and Hundreds, of Fathoms, in the Approximate-Height: (And, if you ſhould want for Thouſands of Fathoms, it is only removing the Decimal-Point in Hundreds of Fathoms, by the Rules of Decimals:) Find the Sum of theſe ſeveral Parts of the Correction, and call it N.

To find that Part of this Equation, called *n*, which correſponds to the Degrees in the Temperature of the Air, which are over and above the Decades, obſerve, that the Correction for 1 Fathom, at the ſeveral Degrees, is here annexed.

Degrees.	1	2	3	4	5	6	7	8	9
Equation for 1 Fathom.	.002	.004	.007	.009	.011	.013	.015	.018	020

Therefore, to find *n*, we have only to multiply the Equation for 1 Fathom, taken from this little Table, by the Number of Fathoms in the Approximate-Height, and the Product will give *n*.

If the Sign of the Correction, N, found by Table II. is +, add the Sum N+*n* to the Approximate-Height; but, if the Sign of Correction, N, is —, ſubtract their Difference, *viz.* N—*n*, from the Approximate-Height, for the correct Height of the Object, which was required.

Example. Let the Approximate-Height be, as found by the laſt Article, 690.543, and the Height of Mercury in the Thermometers, out of Doors, 42 and 57; required, the true Height of the Hill.

Thermometers Heights $\left.\begin{matrix}42\\57\end{matrix}\right\}$

Sum 99

$\frac{1}{2}$

The Temperature of Air $49\frac{1}{2} = 40° + 9\frac{1}{2}$ d.

In Table II. in Column 40+, the Equation is 0; therefore N = +0. In the little Table, in this Art. the Equation for 1 Fathom, correſponding to 1 Degree, is = .002; and therefore to $\frac{1}{2}$ a deg. = .001

Under 9 d. = .020

Hence $9\frac{1}{2}$ d. = .021 this

× by Number of Fathoms, 690, gives 14.49 = *n*. ∴ the whole Equation N+*n* = 0+14.49 = 14.49 Fathoms.

	Fathoms.
To the approximate Height	690.543
Add Equation for the Air N+*n* =	14.490
The Sum is the required Height =	705.033

132. Example 2. Let the Heights of the Mercury in the Barometers and Thermometers be as under:

Barometer.	Therm. in.	Diff.	Therm. out.	Temp. of Air.
Lower 29.45	+38	3	+31	33
Upper 26.82	+41		+35	

Log. of 29.45 14690853
Log. of upper 26.82 14295908

Their Diff. divided by 1000 = 394.945 the groſs Height in Fathoms.

Diff. of *Therm. in* being 3 Deg. gives Equation of Mercury, by *Table* I. } +1.356

Approximate Height 396.301 Fathoms.

K Now,

Now, to find N, we have given the Temperature of the Air $= 33° = 30° + 3°$.

By Table II. 30d. upon 300 Fathoms give 6.683
Upon 90 ——— 2.005
Upon 6 ——— 0.134

Sum is N = 8.822

To find *n*. By the little Table in Art. 131, under 3 Degrees, we have for 1 Fathom .007, this ×d by 396 gives $n = 2.772$. Now, N being — and *n* always +, the Equation is the Difference of theſe Numbers, (*viz.*) $8.822 - 2.772 = 6.050$.

From the approximate Height 396.301 Fath.
Subtract Equation for Air $N - n =$ 6.050

Gives the correct Height in Fathoms 390.251

133. Example 3. Let the Heights of the Barometers be 30.0 and 29.9 Inches, and the Heights of the Thermometers each +40 of *Fahrenheit*'s Scale.

Log. of 30.0 = 14771212
Log. of 29.9 = 14756712

Their Diff. ÷ by 1000 = 14500 Fathoms; this, multiplied by 6, gives 87 Feet; the Height anſwering to $\frac{1}{10}$ of an Inch Fall of the Mercury, in a mean State of the Air; *viz.* when the Thermometer ſtands at 40 d. and the Barometer at 30 Inches.

134. TABLE I.

Deg. of *Bird's Fahrenheit.*	Correction of Dif. of Log.	Fath. or $\frac{1}{1000}$ Parts of Diff. of Log.
1	452	0.452
2	904	0.904
3	1355	1.355
4	1807	1.807
5	2259	2.259
6	2711	2.711
7	3163	3.163
8	3614	3.614
9	4066	4.066
10	4518	4.518
20	9036	9.036
30	13554	13.554
40	18072	18.072
50	22590	22.590
60	27108	27.108
70	31626	31.626

TABLE II.

Deg.	40+	50+ 30—	60+ 20—	70+ 10—	80+ 0—
1	0	0.022	0.044	0.067	0.089
2	0	0.044	0.089	0.134	0.178
3	0	0.067	0.134	0.200	0.267
4	0	0.089	0.178	0.267	0.356
5	0	0.111	0.223	0.334	0.445
6	0	0.134	0.267	0.401	0.535
7	0	0.156	0.312	0.468	0.624
8	0	0.178	0.356	0.535	0.713
9	0	0.200	0.401	0.601	0.802
10	0	0.223	0.445	0.668	0.891
20	0	0.445	0.891	1.337	1.782
30	0	0.668	1.337	2.005	2.673
40	0	0.891	1.782	2.673	3.564
50	0	1.114	2.228	3.342	4.455
60	0	1.337	2.673	4.010	5.347
70	0	1.559	3.119	4.678	6.238
80	0	1.782	3.564	5.347	7.129
90	0	2.005	4.010	6.015	8.020
100	0	2.228	4.455	6.683	8.911
200	0	4.455	8.911	13.367	17.822
300	0	6.683	13.367	20.050	26.733
400	0	8.911	17.822	26.733	35.645
500	0	11.139	22.278	33.417	44.556
600	0	13.367	26.733	40.100	53.467
700	0	15.594	31.189	46.783	62.378
800	0	17.822	35.645	53.467	71.289
900	0	20.050	40.100	60.150	80.209

BOOK III. CHAP. II.

Longimetry.

135. LONGIMETRY ſhews how to apply the Doctrine of Triangles to find the Diſtances of Objects.

Prop. 1. Standing on an Object of a known Height, to find its Diſtance from another Object.

Pl. V. Fig. 5. Example. The Eye being at the Window of the Houſe C, 20 Feet above the horizontal Plane AB, and the ∠ of Depreſſion ACD (DC being conceived to be ∥ to AB) 25 d. required the Diſtance of the Tree from the Houſe, *viz.* AB.

Conſtruction. Make ∠ BAC = 25 d. (for ∠ BAC = ∠ DCA, by the Nature of ∥ s,) and conſtruct the △ as in Caſe 3 of right-angled △s. Then, by that Caſe, we have,

Calculation, As Sine of ∠ A 25 d. : BC 20 Feet ∷ Coſine of ∠ A : AB = 42.89 Feet. *Q. E. I.*

136. *Scholium* 1. In a ſimilar Manner may be found the Depth of a Well, which is of the ſame
Pl. V. Fig. 6. Bigneſs at Top and Bottom. For, meaſuring AB, the Width of the Well, and taking the ∠ of Depreſſion BAC, we have, As Coſine of ∠CAB : AB ∷ S.∠CAB : BC, the Depth which was required.

137. *Scholium* 2. In Order to take the horizontal ∠s, the Quadrant may be ſupported by a Ball and Socket, fixed to the Middle of the under Side, and the Whole ſupported by the Legs of a common Surveying-Table. But as, in Practice, it may frequently happen, that the ∠ to be taken may be obtuſe, it will be better to make Uſe of a Semicircle, inſtead of a Quadrant, which may be thus made.

Pl. V. Fig. 7. Let ADB be a Semicircle of Braſs, of a Radius ſufficient to be divided into Degrees and Minutes; which being carefully done, and numbered both Ways, *viz.* from A towards B, and from B towards A, (by the Method ſhewn in *Geometry*, Art. 211,) on the Center C place a ſtrait Rule to move round the Center, and at the End E a perpendicular open Sight, with a vertical Catgut-String or Horſe-Hair; and at the End F an Eye-Sight, having a vertical Slit for the Eye to look through: At G may be placed a Compaſs; but, if the Poſition with Reſpect to the Meridian be not required, and it is only required to find the Diſtances of Objects, the Compaſs is not abſolutely neceſſary.

138. This Inſtrument being properly ſupported on the three-legged Staff, the Manner of uſing it, to take an ∠, is very eaſy: *Viz.* if it is required to take an ∠ ICH, lay the Index, or Rule, ſo that the middle Line of the Index may correſpond with the Line AB, and through the Sight you ſee the Object I; then, turning the Index about till, through the Sights, you ſee the Object H, the Number of Degrees, in the Arc BE, is manifeſtly = the ∠ required. In like Manner, the oblique Angle KCH may be taken, and the Number of Degrees, in the Arc AE, ſhews the required ∠.

139. If the Objects are at a conſiderable Diſtance, it would be proper to have a Teleſcope inſtead of plain Sights, alſo a Spirit-Level, &c. But as, when we treat of Surveying, we ſhall be obliged fully to deſcribe a Theodolite, an Inſtrument much better adapted to this Purpoſe, when properly conſtructed, we judge it improper to ſay any Thing farther on this Subject here, and therefore paſs on to

140. *Propoſition* 2. To find the Diſtance of two Objects, which are inacceſſible from each other, but both acceſſible from another Place, A.

Example. Wanting to find the Diſtance of two Towers, B, C, at a convenient Diſtance A. I took the ∠ contained BAC = 40 d. then I meaſured AC, which was 2.72 Miles, and AB 2.5 Miles: Required the Diſtance aſunder of the Towers, B, C. Pl. V. Fig. 8.

This is the ſame as Caſe 3 of oblique Triangles; by which BC will be found = 1.797 Mile.

141. *Propoſition* 3. To find the Diſtance of an inacceſſible Object, or Breadth of a River.

Example. Wanting to find the Breadth of a River, I obſerved a Tree cloſe to the other Side: At a convenient Station A, by the Bank of the River, I took the ∠ CAB = 50 d. then meaſuring AB = 100 Yards, I took the ∠ CBA = 60 d. Quære the ⊥ Breadth of the River. Pl. V. Fig. 9.

Conſtruction. Draw $AB = 100$ Yards, make $\angle A = 50$ d. and $\angle B = 60$ d. then is the Place of Interſection, C, that of the Tree on the oppoſite Bank of the River.

Calculation. Firſt $50^\circ + 60^\circ = 110^\circ$, and $180^\circ - 110^\circ = 70^\circ = \angle C$. Then, As S.$\angle C\,70^\circ$: AB 100 Yards :: S.$\angle B\,60^\circ$: AC 92.16 Yards, the Diſtance of the Object C from the Place A, required.

Again, letting fall the $\perp$ CD, we have, in the right-angled $\triangle$ ADC, As Radius 90° : AC 92.16 Yards, :: S.$\angle A\,50^\circ$ :: DC $= 70.59$ Yards. *Q.E.I.*

142. *Scholium.* By this Propoſition it is that the Trees, where Bees hive, are diſcovered in *New-England.* For, in the *Philoſophical Tranſactions*, Number 367, *Jan.* 1721, *Paul Dudley*, Eſq. remarks to this Purport. The Hunter, in a clear ſun-ſhiny Day, takes a Plate or Trencher, with a little Sugar, Honey, or Melaſſes, ſpread on it, and, when got into the Woods, ſets it down on a Rock or Stump there; this the Bees ſoon find out; for it is generally ſuppoſed a Bee will ſcent Honey or Wax above a Mile's Diſtance; and it is obſerved, when one Bee goes Home from the Sugar-Plate, he returns with a conſiderable Number from the Hive. The Hunter ſecures in a Box, or other Conveniency, ſome of them, as they fill themſelves, and, after a little Time, lets one of them go, obſerving very carefully the Courſe it ſteers, by a pocket Compaſs; for, after it riſes in the Air, it flies directly to the Tree where the Hive is. Then the Hunter walks away (as nearly as he can at Right-Angles to the Courſe of the Bee) a convenient Diſtance, which he meaſures; then he lets another Bee fly away, and obſerves, by the Compaſs, his Courſe towards the Tree alſo: Hence he knows, by this Propoſition, the Diſtance the Tree is from him, and, conſequently, the Courſe being alſo known, he can find whereabouts the Tree is, and ſo come at the Honey.

143. *Propoſition* 4. To find the Diſtance of two inacceſſible Objects.

Example. Wanting to find the Diſtance of two Churches, ſeparated from me by a River, I fix on two convenient Stations A, B: Then, at A, I take $\angle BAD = 110°$ and $\angle BAC = 59°$, then meaſuring $AB = 1$ Mile, at B, I take the $\angle ABC = 106°$ and $\angle ABD = 54° 30'$. Pl. V. Fig. 10.

Conſtruction. Draw $AB = 1$ Mile; make the ∠s as in the Example; then will the Places of Interſection, D and C, be the Places of the two Churches, whoſe Diſtance aſunder is required.

Calculation. The $\angle DAB\ 110° + \angle ABD\ 54° 30' = 164° 30'$, and $180° - 164° 30' = 15° 30' = \angle ADB$. Now, as $S. \angle ADB\ 15° 30' : AB\ 1$ Mile $:: S. \angle ABD\ 54° 30' : AD\ 3.046$ Miles. Again, the $\angle ABC\ 106° + \angle BAC\ 59° = 165°$. $\therefore \angle ACB = 180° - 165° = 15°$. Hence, by Trigonometry, As $S. \angle ACB\ 15° : AB\ 1$ Mile $:: S. \angle ABC\ 106° : AC\ 3.714$ Miles. Now, in the Triangle DAC, we have given $DA = 3.046$ Miles, $CA = 3.714$ Miles, and the ∠ contained DAC $(= \angle DAB\ 110° - \angle BAC\ 59° =)\ 51°$; which is the 3d Caſe of oblique Triangles, by which DC will be found $= 2.971$ Miles. *Q. E. I.*

144. In the ſame Manner the Diſtances of three or four Objects may be found. When, by theſe Means, the Diſtance of two Towers or Objects is found, that may be taken for a new Baſe; and, by ſuch Means, others may be found, and ſo a Chain of Triangles formed; by which Means, if the ∠s are carefully taken, by Inſtruments accurately divided, the Situation of the principal Towers, Hills, *&c.* in a Country, may be taken.

145. *Propoſition* 5. To find the Diſtance and Poſition of a Place from three other Places, whoſe Poſition and Diſtances from each other are given.

This Propoſition admits of 6 Caſes; moſt of which are of frequent and great Uſe in Surveying, as will appear by the Examples.

Caſe 1. When the Station is without the Triangle made by the three given Objects, but in a Line with one of the Sides produced.

Example. Having, in a Map of an Eſtate, found Pl. V. Fig. 11. three Trees, A, C, B, planned, but the Tree D in a direct Line with the Trees A, C, I found, from A to C meaſured 8.90 Chains, A to B 12.20, and B to C 7.70; and that the ∠ CDB was 27°. Quære the Diſtance AD, in Order to ſet the Tree in its proper Place in the Map.

Conſtruction. The △ ABC being formed, (and the ∠ BAC being* = the two internal and oppoſite ∠s D and DBA) make the ∠DBA = the Difference of the ∠s BAC and BDA, then will BD cut CA, produced, in D, the Point required.

* 70 g.

Calculation. The three Sides of the Triangle ACB being given, find the ∠ BAC by Caſe 4 of oblique Triangles; which, in the above Example, will be found = 39° 0′: Then ∠ BAC 39° 0′ — ∠ADB 27° = 12° = ∠DBA, for the Reaſon given in the Conſtruction. Hence, in △ DAB, we have, As S.∠D : AB :: S.∠DBA : DA, =, in this Example, 5.587 Chains. *Q. E. I.*

146. *Caſe* 2. When the Station required is in one of the Sides of the given △.

Pl. V. Fig. 12. Example. Let the △ABC be the ſame as in the firſt Example: And, having Occaſion to lay down an Object at D, I took the ∠ BDC = 73°. Quære the Diſtance DB or DA.

Conſtruction. Make ∠ BAE = the obſerved ∠ BDC, in our Example, = 73 d. draw CD ∥ EA; then is D manifeſtly the Place required.

Calculation. In the △ ABC, by the 4th Caſe of oblique Triangles, find the ∠ B; then, the ∠s B and BDC being known, the ∠ DCB is alſo known. Hence we have, As S.∠BDC : BC :: S.∠DCB : BD, equal, in the above Example, to 6.996 Chains.

147. *Caſe* 3. When the three given Places are in a Right-Line.

Pl. V. Fig. 13. EXAMPLE. Being at Sea, near a ſtrait Shore, I obſerved three Objects, A, B, C, which were truly laid down on my Chart; but, wanting to lay down the Place of a ſunken Rock, D, by *Hadley*'s* Octant, I took the ∠ADB = 26 d. 15′ and ∠BDC = 23 d. 30′, and, by meaſuring on the Map, found AB = 2.2 Miles, BC = 1.55 Mile.

Conſtruction. On AB deſcribe § the Segment of a § 257. g. Circle, ADB, capable of containing an ∠ = 26 d. 15′, and, on BC, the Segment BDC, capable of containing an ∠ = 23 d. 30′: Then, as the Point D muſt be Pl. V. Fig. 14. ſomewhere in the Arc of the Segment ADB, and alſo ſomewhere in the Arc of the Segment CDB, it muſt be where they interſect in D: Therefore, joining D, A, D, B, and D, C, it is manifeſt DA, DB, and DC, are the Diſtances required.

Calculation. Having drawn the Radii AE, ED, EB, alſo the Radii FD, FB, FC, from the Middle of BD, *viz.* G, draw Lines GE, GF: Now, the 3 Sides of the Triangles EDG, EBG, being reſpectively equal, (*viz.* ED = EB, BG = DG, by Conſtruction, and EG common,) theſe △s muſt be equal to each other in every Reſpect; and ∵ ∠EGD = EGB; and ∵ each a Right-∠. For the like Reaſon, the ∠ DGF is a Right-∠; and, conſequently, EGF, joining the Centers of the two Circles, is a Right-Line. Let fall the ⊥s EI, FH; then, the ∠ at the Center being double ‡ the ∠ at the Circumference, the ‡ 145. g. ∠AEB = twice the ∠ ADB, ∵ Half the ∠ AEB, *viz.* IEB = ∠ADB = 26 d. 15′; ∵ ∠IBE = 90° — ∠IEB 26 d. 15′ = 63 d. 45′; and, for the ſame Reaſon, ∠ BFH = ∠BDC = 23 d. 30′, and ſo ∠ FBH = 90 d. — 23 d. 30′ = 66 d. 30′; but the ∠IBE + ∠EBF + ∠FBH = 180 d. ∵ ∠EBF = 180 d. — ∠IBE 63 d. 45′ — ∠FBH 66 d. 30′ = 49 d. 45′.

In the right-∠ △ IBE are now given IB = Half AB = 1.1 Mile; and ∠s to find EB, the Radius of

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* The Manner of taking ſuch Angles, by *Hadley*'s Octant, will be ſhewn in our *Elements of Navigation*, which are nearly ready for the Preſs.

the Circle ABD, =2.487 Miles. Alſo, in the right-∠d △ BFH are given BH=Half BC =0.775, and ∠s to find BF, the Radius of the Circle BCD, = 1.944.

Hence, in the △ EBF, we have now given EB = 2.487, BF=1.944, and contained ∠ EBF=49 d. 45′ (as found above) to find the ∠ BEF = 50 d. 19′.

Then, in the right-∠ △ BEG, are given BE and BEG to find BG = 1.914 Miles. But BG = Half BD by Suppoſition; ∵ BD = twice BG = 3.828 Miles.

Again, the ∠ BAD being at the Circumference, and the ∠ BED at the Center, of the ſame Circle BAD, and ſtanding on the ſame Baſe BD, the ∠ BAD is = Half* the ∠ BED, *viz.* = ∠ BEG = 50d. 19′ above found. Hence, in the △ BAD are given ∠s A and D, and conſequently the ∠ B=103d. 26′, alſo the Side AB or BD to find the Side AD = 4.838.

* 145. g.

Laſtly, in the Triangle ADC are given the Sides AD, AC, (=AB+BC,) and ∠ CAD (BAD) to find DC=3.781 Miles.

Or thus.

Pl. V. Fig. 15.

148. *Conſtruction.* Make ∠ ACE = ∠ ADB = 26 d. 19′; and ∠ CAE = ∠ BDC = 23 d. 30′; and, from the Point of Interſection E, through B, draw a Line to ED to interſect the Arc ADC; join A, D, and D, C; then are DA, DB, DC, the required Diſtances.

Demonſtration. For the ∠s ACE and ADE (ADB) ſtand on the ſame Baſe, AE, in the ſame Segment ADCE, and are therefore equal, by Art. 146 of our *Geometry.* Alſo the ∠ EAC and ∠ EDC are equal, being on the ſame Baſe, EC, of the ſame Segment EADC.

Calculation. In Triangle AEC are given ∠s EAC, ECA, and conſequently ∠ AEC; alſo AC to find AE= 2.173 Miles. Then, in △ ABE will be given two Sides AB, AE, and contained ∠ BAE = 23 d. 30′, to find ∠s AEB = 79 d. 57′ ABE = 76 d. 33′.

But ∠DBC = ∠ABE, by Art. 69, *Geometry.* Hence, in △ BDC are given ∠D and ∠DBC, and conſequently ∠BCD is alſo known; alſo BC is given by the Queſtion, whence may be found BD = 3.828 and DC = 3.781.

Laſtly, in △AED are given ∠s ADE, AED, and Side AE, to find AD = 4.838.

149. *Caſe* 4. When the three Objects form a Triangle, and the Station is without the Triangle.

EXAMPLE. Wanting to determine the Poſition of a Rock, at a Diſtance from the Shore or main Land, by *Hadley*'s Octant, I took the ∠ ADC = 54 d. ∠ CDB = 33 d. and, by meaſuring on an actual Survey of the Coaſt, I found AC = 2.27 Miles, BC = 1.55 Miles, and AB = 3.3 Miles. Hence it is required to find the Diſtance of each of the 3 Places from the Station D.

Pl. VI. Fig. 1.

The *Conſtruction* is nearly the ſame as in the laſt; *viz.* on AC deſcribe a Circle capable of containing an ∠ ADC = 54 d. and on CB a Segment capable of containing an ∠ CDB of 33 d. then is the Point of Interſection, D, the Place required.

Pl. VI. Fig. 2.

Calculation. Conceive the ſeveral Radii, *&c.* to be drawn as in the Figure: Then, in △ ABC are given three Sides to find ∠ACB = 118 d. 17′: And, for the Reaſon ſhewn in the laſt Caſe, in the right-∠d △ CEF are given EC = Half AC = 1.135, and ∠ EFC = ∠ADC = 54 d. to find the Radius FC = 1.403. Alſo, in the right-∠ △ CHG are given CH = Half CB = 0.775, and ∠ CGH = ∠ CDB = 33 d. to find the Radius GC = 1.423.

Now, the ∠ ECF = 90 d. — ∠EFC 54 d. = 36 d. and ∠ GCH = 90 d. — ∠CGH 33 d. = 57 d. But ∠ACB 118 d. 17′ — ∠ECF 36 d. — ∠ GCH 57 d. = ∠FCG = 25 d. 17′. Hence, in the △ FCG are given the two Sides FC, GC, and ∠ contained FCG, to find ∠FGC = 75 d. 33′ and ∠ GFC 79 d. 10′. Now, in the right-∠ △ GIC are given GC and ∠ IGC (FGC) to find IC = 1.378, which, doubled, gives DC = 2.756. Again, ∠CGB being = twice

∠CGH = 66d. and ∠CGD being = twice ∠ CGI = twice 75 d. 33′ = 151 d. 16′, the ∠DGB = 360d. — ∠CGB 66 d. — ∠CGD 151d. 16′ = 142d. 44′. Hence, Sum of the ∠s GDB and GBD = 180 d. — ∠DGB 142 d. 44′ = 37 d. 16′. But ∠s GDB and GBD are =; ∵ each = Half of 37d. 16′ = 18d. 38′.

Now, in △ DGB we have, As S.∠B 18 d. 38′ : GD (GC) 1.423 :: S.∠DGB 142d. 44′ : DB = 2.697.

In the ſame Manner, ∠AFD may be found = 360 d. — ∠ AFC 108 d. — ∠ CFD 158 d. 20′ = 93 d. 40′, and 180 d. — ∠AFD = Sum of the ∠s FAD and FDA, which, being equal, will each be = Half that Sum, *viz.* 43 d. 10′ : ∵ in △ AFD we have, *laſtly*, As S.∠ADF 43d. 10′ : AF (FC) :: S.∠AFD : AD = 2.047.

Or thus.

150. *Conſtruction.* Make ∠EBA = ∠ ADE 54d. and ∠ BAE = ∠ EDB = 33 d. Then, through the Points A, B, and the Interſection E, deſcribe a Circle AEBDA; through E, C, draw EC and produce it to interſect the Circle at D; join AD, BD; then will the Diſtances AD, CD, BD, be thoſe required. For ∠s BAE and BDE, ſtanding on the ſame Baſe, BE, in the ſame Segment BDAE, are equal, and alſo ∠s ABE and ADE, ſtanding on the ſame Chord, AE, in the ſame Segment ADBE, are equal.

Pl. VI. Fig. 3.

Calculation. Firſt, in △ ABC are given three Sides to find ∠BAC = 24d. 26′. Now, in △AEB are given ∠ BAE = 33 d. ∠ ABE = 54 d. and ∠ AEB = 93 d. and Side AB = 3.3 to find the Sides AE = 2.674 and BE = 1.8. Then, in △ ACE will be given the two Sides AE and AC 2.27, and their contained ∠ EAC = ∠BAE 33 d. — ∠ BAC 24 d. 26′ = 8 d. 34′, to find ∠AEC = 38 d. 19′.

Again, in △ AED are given AE = 2.674, ∠ AED = 38d. 19′, ∠ADE = 54d. and conſequently ∠DAE = 87 d. 41′ to find AD = 2.049. Then

∠

∠ ADE 54 d. + ∠ AEC 38 d. 19′ = 92 d. 19′, and 180 d. — 92 d. 19′ = 87 d. 41′ = ∠ DAE. But ∠ EAD 87 d. 41′ — ∠ EAC 8 d. 34′ = ∠ CAD = 79 d. 7′. Hence, in △ ACD are given AC = 2.27, ∠ ADC = 54 d. and ∠ CAD = 79 d. 7′, to find DC = 2.755.

Laſtly, ∠ AEB 93 d. — ∠ AEC 38 d. 19′ = ∠ DEB = 54 d. 41′. Hence, in △ DEB are given EB = 1.8, ∠ DEB = 54 d. 41′, and ∠ EDB = 33 d. to find BD = 2.696.

N. B. In this Method of conſtructing, when the ∠ BDC is leſs than the ∠ BAC, the Point C will be above the Point E; but the Calculation will be ſo exactly like the above as to require no particular Explanation. When the Points E and C happen to fall too near together to produce EC towards D with Certainty, the firſt Method of Conſtruction will be more accurate.

151. *Caſe* 5. When the Point D, or our Station, falls without the △ ABC, but the Point C falls towards D.

Example. Let A, B, C, repreſent three Towers, Pl. VI. whoſe Diſtances from each other are known: *Viz.* Fig. 4. AB = 3.3 Miles, AC = 2.27 Miles, and BC = 1.55 Miles. On another Tower, D, I took the ∠ ADC = 34 d. and ∠ BDC = 17 d. 30′. It is required to find the Diſtance of the Tower D from each of the other Towers.

Conſtruction. On AC and BC deſcribe Segments Pl. VI. of Circles capable of containing Angles of 34 d. and Fig. 5. 17 d. 30′ reſpectively; then will the Place of their Interſection, D, be the Point required, as in the laſt Caſe.

The Learner muſt, by this Time, be able to ſolve the common Caſes of Plane Trigonometry, and therefore we may now ſave ourſelves the Trouble of computing the Numbers, it being ſufficient, for the Future, only to explain the Method of Solution, leaving the actual Computation for his own Exerciſe.

Method of Calculation. Conceive the ſeveral Radii, *&c.* to be drawn as in the Figure: Then find the ∠ ACB, the Radii GC, and FC, as ſhewn in the laſt Caſe. Alſo the ∠s FGC and GFC, alſo CD.

Now the ∠ CGB is = twice the ∠ CDB by the Conſtruction; alſo the ∠ CGD = twice the ∠ CGI (CGF) CG being = GD, and therefore the △ CGD Iſoſceles. But, by the Figure, ∠BGC + ∠CGD = ∠BGD; and therefore, as the △ BGD is Iſoſceles, the ∠GBD = ∠GDB, and each = to $\frac{180d. - \angle BGD}{2}$. Hence, by Trigonometry, BD may be found, by ſaying, As S.∠DBG : GD :: S.∠BGD : BD. In like Manner AD may be found. We have been but ſhort in explaining this, becauſe, if the Learner underſtands the laſt Caſe, what is here ſaid will be ſufficient.

Otherwiſe thus.

152. This Caſe may be conſtructed exactly like the ſecond Method given in the laſt Caſe, and then the Calculation will be as follows.

Pl. VI. Fig. 6. Firſt, in the △ ABC are given the three Sides to find the Angle BAC.

Secondly, in the △ AEB are given AB and ∠s EAB, EBA, and conſequently the ∠ AEB, to find AE and EB.

Thirdly, in the △ ACE are now given AE, AC, and the contained ∠ EAC, (= ∠BAC + ∠EAB,) to find the ∠ AEC.

Fourthly, in the △ ADE are given AE, ∠ AED, (AEC,) and ∠ ADE, to find AD.

Fifthly, in the △ AED, the ∠s AED and ADE being known, the ∠ EAD is alſo known. But the ∠EAD − ∠EAC = ∠CAD. Hence, in the △ CAD are given AC, and ∠s CAD, ADC, to find DC.

Sixthly and laſtly, the ∠ AEB − ∠ AEC = ∠ DEB. Hence, in the △DEB are given EB, the ∠s DEB, and EDB, to find BD.

153. *Scholium.* This Caſe is ſo much like the laſt, that perhaps ſome of our more learned Readers may think we might have contented ourſelves with the fourth Caſe only; but, as we are writing an Introduction, it ſeems more excuſable to ſay rather too much than too little.

154. *Caſe* 6. When the Station falls within the Triangle made by the given Objects.

Pl. VI. Fig. 7.

Example. Let A, B, C, repreſent three Towers, whoſe Diſtance from each other is known: And, wanting to find the Diſtance from another Tower, D, to each of theſe Towers, I took the Angle ADC = 116 d. ∠ BDC = 110 d. and, to prove the Truth of theſe Angles, I alſo took the ∠ ADB = 134 d. which, together, making 360 d. or a whole Circle, proves them to have been accurately taken.

Pl. VI. Fig. 8.

Conſtruction. On two of the given Sides deſcribe Segments of Circles capable of containing the given Angles: For Example, On AC a Segment to contain an Angle of 116 d. and on AB one to contain an Angle of 134 d. Then will the Point of Interſection, D, be manifeſtly the Place required.

Calculation. Let F and G be the Centers of the two Circles whoſe Segments are deſcribed, and let GH be drawn ⊥ to AC, and FE ⊥ to AB, and join F, G; alſo conceive the Radii to be drawn as in the Figure.

Firſt, in the △ ABC are given the three Sides to find the ∠ CAB.

Secondly, conceive, in your Mind, the Circle of which the Arc ADC is a Part to be completed, and that Lines be drawn from the Points A and C to any Point in the oppoſite Segment, ſo as to form, with the Lines AD, CD, a quadrilateral Figure inſcribed in the Circle; then the ∠ ADC + the Angle in the oppoſite Segment being = 180 d. the Angle in the oppoſite Segment muſt be = 180 d. − ∠ ADC. And $\frac{1}{2}$ ∠ at the Center, *viz.* ∠ AGH, = the Angle in that oppoſite Segment. Hence, in the right-angled

△

$\triangle$ AGH are now given $\angle$ AGH and AH $= \frac{1}{2}$ AC to find the Radius AG.

Thirdly, in like Manner, the Angle AFE may be ſhewn to be $= 180° - \angle$ ADB. Hence, in the right-angled $\triangle$ AEF are given AE $= \frac{1}{2}$ AB, and $\angle$ AFE, to find the Radius AF.

Fourthly, the $\angle$ GAH being $= 90° - \angle$ AGH becomes known; and, in like Manner, the $\angle$ FAE is known. Hence, $\angle$ FAG becomes known, being $= \angle GAH + \angle CAB + \angle FAE$. Hence, in the $\angle$ GAF are now given the Sides AG, AF, and their contained $\angle$ GAF, to find the Angle AGF.

Fifthly, for Reaſons already ſhewn in former Caſes, the Line GF, joining the Centers G and F, divides AD in I into two equal Parts, and is perpendicular thereto: Therefore, in the right-angled $\triangle$ AGI we have now given AG and $\angle$ AGI (AGF) to find AI, which doubled gives AD.

Sixthly, hence, in the $\triangle$ ADC are now given two Sides, AC, AD, and $\angle$ ADC, to find DC.

Seventhly and laſtly, in the $\triangle$ CDB we have now given two Sides, CD, CB, and the $\angle$ CDB, to find DB.

Otherwiſe thus.

Pl. VI. Fig. 9.

155. *Conſtruction.* This may be done in nearly the ſame Manner as the ſecond Method in Caſe 4; *viz.* make the Angle ABE $= \angle$ ADE *viz.* $= 180° - \angle$ ADC; and $\angle$ BAE $= \angle$ BDE *viz.* $= 180° - \angle$ BDC. Then deſcribe a Circle through the three Points A, B, E, and join E, C, by the Line EC; then will the Point D, where the Line EC interſects the Circle EADBE, be the Place of the required Station.

Demonſtration. For, by the Conſtruction, the Angles ABE, ADE, are in the ſame Segment, ADBE, and ſtand on the ſame Chord AE, and therefore muſt be equal, by Art. 146 of our *Geometry.* Alſo the Angles BAE, BDE, are equal to each other, being on the ſame Baſe, BE, and in the ſame Segment, BDAE. But the Angles ADE, BDE, are Supplements of the Angles ADC, BDC; and the Angles ABE, BAE, were made equal to theſe Supplements in the Conſtruction; the Point D is manifeſtly that required.

Calculation. Firſt, in the △ ABC are given the three Sides to find the Angles A and C.

Secondly, in the △ AEB are given AB and all the Angles to find AE.

Thirdly, in the △ CAE are now given CA, AE, and contained Angle CAE, $= \angle CAB + \angle BAE$, to find the Angle ACE.

Fourthly, In the △ ADC are now given AC, and Angles, to find the required Diſtances AD and DC.

Fifthly and laſtly, in the △ CDB are now given BC, DC, and their contained Angle DCB, $= \angle ACB - \angle ACD$, to find DB, the laſt required Diſtance.

156. *Propoſition* 6. Given, the Diſtance of two Objects, A, B, and the Angles ADB, BDC, BCA, ACD, to find the Diſtance of the two Stations, D, C, from the Objects, A, B. Pl. VI. Fig. 10.

Conſtruction. Draw DC any convenient Length, at Pleaſure, and make the ∠s ADB, BDC, BCA, ACD, = the given Angles; and join the Points of Interſection by the Line AB: Then, if AB be made on the Sector = the given Length, DC, DA, &c. may be meaſured on the Sector.

157. A Sector is generally given in a Caſe of Pocket-Inſtruments, and is a very valuable Inſtrument; but, as its Deſcription and Uſe would take up more Room than we can ſpare in this Place, we muſt defer its Deſcription, &c. till ſome other Opportunity: And, therefore, for the Sake of ſuch as are not acquainted with the Uſe of a Sector, it is neceſſary to give another Conſtruction, *viz.*

Make *dc* = any Number, at Pleaſure, and make the Angles *bdc*, *adc*, *acd*, *bca*, reſpectively, = the given Angles BDC, ADC, ACD, BCA; and join *a*, *b*; then it is plain this Figure muſt be ſimilar to that required:

quired: Therefore, having drawn AB = the given Diſtance, make the Angle ABC = ∠ *abc*, and ∠ BAC = ∠ *bac*; alſo the ∠s BAD, ABD, reſpectively, = the Angles *bad*, *abd*; then will the Points of Interſection, D, C, be thoſe required. Join D, C.

Calculation. In the △ *adc* are given *dc* = the aſſumed Number, the Angle *adc* = *adb* + *bdc*, and Angle *acd*, and conſequently the remaining Angle *dac*, to find *ad*, *ac*.

In like Manner, in the △ *bcd*, are given the three Angles and the Side *dc* to find *db*, *bc*.

Now, in the △ *adb* are given the two Sides *ad*, *bd*, and contained Angle *adb*, to find *ab*. Hence, by the Nature of ſimilar Figures, we have, As *ab* : AB :: *dc* : DC, and :: *ad* : AD, :: *ac* : AC, :: *bd* : BD, and :: *bc* : BC.

158. *Propoſition* 7. Given, the Diſtances of 3 Pl. VI. Fig. 11. Objects, A, B, C, from each other, the ∠s ADC, CDE, CED, and CEB, to find the Sides AD, DC, DE, EC, and EB.

Conſtruction. Aſſume any Line, *de*, at Pleaſure; Pl. VI. Fig. 12. make the Angle *cde* = ∠ CDE, and Angle *ced* = ∠ CED; alſo the Angle *cda* = ∠ CDA, and ∠ *ceb* = ∠ CEB. Produce *ad* and *be* to interſect each other at *f*, and join *cf*.

Then, it is manifeſt, the Figures *cdfe*, CDFE, are ſimilar. Therefore, on AC deſcribe a Segment of a Circle capable of containing an ∠ AFC = ∠ *afc*; and on CB a Segment capable of containing an Angle CFB = ∠ *cfb*. From the Point of Interſection, F, draw FA, FB, FC. Make the Angle FCD = ∠ *fcd*, and ∠ FCE = ∠ *fce*; which completes the Conſtruction.

Calculation. Aſſume *de* = any Number; then, in the △ *def* are given *de*, ∠ *fde* = 180° − the Sum of the Angles *adc*, *cde*, and ∠ *def* = 180° − the Sum of the ∠s *bec*, *ced*, and conſequently the remaining ∠ *dfe*, to find *df* and *fe*. Alſo, in the △ *dce*, the Angles and Side *de* are given, to find the Sides *dc* and ce. Hence, in the $\triangle\, dcf$ are now given dc, df, and contained $\angle\, cdf$, to find $\angle\, dfc$. Hence alſo, $\angle\, cfe$ becomes known, being $= \angle\, dfe - \angle\, dfc$. We have now given AC and Angle AFC, alſo BC and Angle BEC, which reduces the Problem now to the ſame as Prop. 5, Caſe 4, by which may be found FA, FC, FB: Then, by ſimilar Triangles, As fc : cd :: FC : CD; and as fc : ce :: FC : CE.

Now, in the $\triangle$ ACD are given AC, DC, and $\angle$ ADC, to find AD.

Laſtly, in the $\triangle$ CBE are given CB, CE, and Angle BEC, to find BE.

159. *Scholium.* This Solution fails when AD is parallel to BC; or, which is the ſame, when the Angle ADE + BED is $= 180°$. For, in ſuch Caſe, from the Nature of Parallels, AD and BE, produced, could never meet. For which Reaſon, it may be proper to give another Method of Solution.

Conſtruction. Having deſcribed the Segments ADC, BEC, to contain the given Angles ADC and BEC, reſpectively, draw the Line CF to cut off a Segment = Angle CDF, and the Right-Line CG to cut off a Segment = Angle CEG; then, it is manifeſt, the Points G, F, muſt be in the Right-Line DE: Therefore, join G, F, and produce the Line each Way till it interſects the Segments, then will the Points of Interſection, D and E, be the Stations required.

Pl. VI. Fig. 13.

Calculation. Conceive the Radii and Perpendiculars drawn, as in the Figure is ſhewn by the dotted Lines; then, by Trigonometry, in the right-angled $\triangle$ CIH, we have, As S. $\angle$ IHC ($=$ Angle ADC) : IC ($\frac{1}{2}$ AC) :: R : HC.

Secondly, in the right-angled $\triangle$ CKH, As R : HC :: S. $\angle$ KHC ($=$ Angle CDF) : CK. Then $2\,\mathrm{CK} = \mathrm{CF}$.

Thirdly, in like Manner, by the other Segment, may be found CG.

Fourthly, the Angle $ICH = 90° - \angle IHC$; and $HCK = 90° - \angle KHC$; then Angle $ICH + \angle HCK = \angle ACF$. In like Manner, the Angle BCG becomes known, being equal to the Angle $MCL + \angle LCN$. But the Angle $ACF + \angle BCG - \angle ACB = $ [*] $\angle GCF$.

Fifthly, hence, in the Δ GCF are now given GC, FC, and the contained Angle GCF, to find the other Angles. Hence, the Angles CGD, CFE, become known: And, the Trapezium ACFDA being in a Circle, the Angle $DAC + \angle DFC = 180°$, and conſequently Angle $DAC = 180° - \angle DFC$. In like Manner the Angle EBC becomes known.

Sixthly, hence, in the Δ DAC we have now given AC, and Angles, to find DC and DA.

Seventhly, in like Manner, in the Δ EBC are given BC, and the Angles, to find EC and EB.

Eighthly and laſtly, in the Δ DCE are now given DC, EC, and all the Angles, to find DE; and ſo the Problem will be completely ſolved.

160. *Scholium.* From this Conſtruction it appears, that, when the Points G and F coincide, the Propoſition is indeterminate; as, in ſuch Caſe, the Line DE may be drawn in any Direction.

On my communicating this Problem to a Friend, he brought me a Solution nearly the ſame, in Subſtance, as the ſecond Method here given.

161. If Room would have permitted in the Plate, I intended to have given other Methods of Solution. Would Room admit, I might alſo have added ſeveral other Problems; but theſe are ſufficient to ſhew

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[*] If the Truth of this Equation ſhould not be ſufficiently plain, it may be ſhewn thus:

For the $\angle ACG + \angle GCF = \angle ACF$
And $\angle BCF + \angle GCF = \angle BCG$

Sum $\angle ACG + \angle BCF + 2\angle GCF = \angle ACF + \angle BCG$:
But $\angle ACG + \angle BCF + \angle GCF = \angle ACB$.

By Subtraction, the $\angle GCF = \angle ACF + \angle BCG - \angle ACB$, the ſame as above.

the great Uſefulneſs of Trigonometry in taking Heights and Diſtances of Places, and in ſurveying of Counties, Sea-Coaſts, and Harbours; and, if the young Student will make himſelf Maſter of theſe, it may be ſuppoſed he will not meet with any Difficulties, in real Practice, which he cannot readily overcome, as very few of our Surveyors even take the Pains to underſtand thus much. However, we may, perhaps, hereafter, take another Opportunity to add ſome more Problems, either in our intended Treatiſe of Surveying, or in a Treatiſe to be entitled *Trigonometrical Eſſays.* In which, if Time, Health, and Encouragement permit, I propoſe to ſhew, firſt, A Method of conſtructing Tables of natural and artificial Sines and Tangents, &c. with the Conſtruction of Scales of Sines, Tangents, &c. as hinted at in Art. 26 of this Eſſay. Secondly, The Conſtruction and Uſe of the Sector, &c. Thirdly, The Method of ſolving Triangles, independent of Tables or Inſtruments. Fourthly, The algebraical and geometrical Solutions to a Variety of Queſtions; with the Method of deducing geometrical Conſtructions from algebraical Solutions, &c.

162. Agreeable to our Promiſe, in Page 12, we ſhall conclude this Eſſay with the following uſeful Table.

A TABLE *of Natural Sines, Tangents, and Secants.*

RADIUS = 1.

D	M	Sine.	Coſine.	Tang.	Co-Tan.	Sec.	Co-Sec.	M	D
0	0	.0000	1.000	.0000	Infinite.	1.000	Infinite.	0	90
	1	.0003	1.000	.0003	3438.	1.000	3438.	59	
	10	.0029	1.000	.0029	343.8	1.000	343.8	50	
	20	.0058	1.000	.0058	171.9	1.000	171.9	40	
	30	.0087	1.000	.0087	114.6	1.000	114.6	30	
	40	.0116	.9999	.0116	85.94	1.000	85.95	20	
	50	.0145	.9999	.0145	68.75	1.000	68.76	10	
1	0	.0174	.9998	.0174	57.29	1.000	57.30	0	89
	10	.0203	.9998	.0203	49.10	1.000	49.11	50	
	20	.0233	.9997	.0233	42.96	1.000	42.97	40	
	30	.0262	.9996	.0262	38.18	1.000	38.20	30	
	40	.0291	.9995	.0291	34.36	1.000	34.38	20	
	50	.0320	.9994	.0320	31.24	1.000	31.26	10	
2	0	.0349	.9994	.0349	28.64	1.000	28.65	0	88
	10	.0378	.9993	.0378	26.43	1.000	26.45	50	
	20	.0407	.9992	.0407	24.54	1.000	24.56	40	
	30	.0436	.9990	.0437	22.90	1.000	22.92	30	
	40	.0465	.9990	.0466	21.47	1.001	21.49	20	
	50	.0494	.9988	.0495	20.21	1.001	20.23	10	
3	0	.0523	.9986	.0524	19.08	1.001	19.11	0	87
	10	.0552	.9985	.0553	18.07	1.001	18.10	50	
	20	.0581	.9983	.0582	17.17	1.002	17.20	40	
	30	.0610	.9981	.0612	16.35	1.002	16.38	30	
	40	.0640	.9980	.0640	15.60	1.002	15.63	20	
	50	.0668	.9980	.0670	14.92	1.002	14.96	10	
4	0	.0698	.9976	.0699	14.30	1.002	14.34	0	86
	10	.0726	.9973	.0728	13.73	1.003	13.76	50	
	20	.0756	.9971	.0757	13.20	1.003	13.23	40	
	30	.0784	.9969	.0787	12.71	1.003	12.74	30	
	40	.0813	.9967	.0816	12.25	1.003	12.29	20	
	50	.0843	.9964	.0846	11.26	1.003	11.87	10	
5	0	.0871	.9962	.0875	11.43	1.004	11.47	0	85
	10	.0901	.9960	.0904	11.06	1.004	11.10	50	
	20	.0929	.9957	.0933	10.71	1.004	10.76	40	
	30	.0958	.9953	.0963	10.39	1.004	10.43	30	
	40	.0990	.9951	.0992	10.08	1.005	10.13	20	
	50	.1016	.9948	.1021	9.788	1.005	9.839	10	
5	60	.1045	.9945	.1051	9.514	1.006	9.567	0	84
		Coſine.	Sine.	Co-Ta.	Tangent.	Co-S.	Secant.		

D.	M.	Sines.	Coſines	Tang.	Co-Ta.	Secants	Co-Se.	M.	D.
6	0	.1045	.9945	.1051	9.514	1.006	9.567	0	84
	10	.1074	.9942	.1080	9.255	1.006	9.309	50	
	20	.1103	.9939	.1109	9.009	1.006	9.065	40	
	30	.1132	.9936	.1139	8.777	1.006	8.834	30	
	40	.1161	.9932	.1169	8.556	1.007	8.613	20	
	50	.1190	.9929	.1198	8.345	1.007	8.404	10	
7	0	.1219	.9925	.1228	8.144	1.007	8.205	0	83
	10	.1247	.9922	.1257	7.953	1.008	8.016	50	
	20	.1276	.9918	.1287	7.770	1.008	7.834	40	
	30	.1305	.9914	.1516	7.596	1.009	7.661	30	
	40	.1334	.9910	.1346	7.428	1.009	7.495	20	
	50	.1363	.9907	.1376	7.269	1.009	7.337	10	
8	0	.1392	.9902	.1405	7.115	1.009	7.185	0	82
	10	.1420	.9899	.1436	6.968	1.010	7.040	50	
	20	.1449	.9894	.1464	6.826	1.010	6.900	40	
	30	.1478	.9890	.1494	6.691	1.011	6.765	30	
	40	.1507	.9886	.1524	6.560	1.011	6.636	20	
	50	.1536	.9881	.1554	6.435	1.012	6.512	10	
9	0	.1564	.9877	.1583	6.313	1.012	6.392	0	81
	10	.1593	.9872	.1614	6.197	1.013	6.277	50	
	20	.1622	.9868	.1643	6.084	1.013	6.166	40	
	30	.1650	.9862	.1673	5.976	1.014	6.059	30	
	40	.1680	.9859	.1703	5.871	1.014	5.955	20	
	50	.1708	.9853	.1733	5.769	1.015	5.855	10	
10	0	.1736	.9848	.1763	5.671	1.015	5.759	0	80
	10	.1765	.9843	.1793	5 576	1.016	5.665	50	
	20	.1793	.9838	.1823	5.484	1.016	5.575	40	
	30	.1822	.9832	.1853	5.395	1.017	5.487	30	
	40	.1851	9827	.1883	5.309	1.017	5.402	20	
	50	.1880	.9822	.1914	5.226	1.018	5.320	10	
11	0	.1908	.9816	.1944	5.144	1.018	5.240	0	79
	10	.1936	.9811	.1974	5.065	1.019	5.163	50	
	20	.1965	.9805	.2004	4.989	1.019	5.089	40	
	30	.1994	.9799	.2035	4.915	1.020	5.016	30	
	40	.2022	9793	.2064	4.843	1.021	4.945	20	
	50	.2051	.9787	.2095	4.773	1.022	4.876	10	
11	60	.2079	.9781	.2125	4.704	1.022	4.809	0	78
		Coſine	Sine.	Co Ta.	Tang.	Co-Se.	Secants		

D.	M.	Sines.	Coſines	Tang.	Co-Ta.	Secants	Co-ſec.	M.	D.
12	0	.2079	.9781	.2125	4.704	1.022	4.809	0	78
	10	.2107	.9775	.2155	4.638	1.023	4.745	50	
	20	.2136	.9769	.2186	4.574	1.023	4.682	40	
	30	.2164	.9763	.2217	4.511	1.024	4.620	30	
	40	.2193	.9757	.2247	4.449	1.025	4.560	20	
	50	.2221	.9750	.2278	4.390	1.025	4.502	10	
13	0	.2249	.9744	.2309	4.331	1.026	4.445	0	77
	10	.2278	.9737	.2339	4.274	1.027	4.390	50	
	20	.2306	.9730	.2370	4.219	1.027	4.336	40	
	30	.2334	.9724	.2401	4.165	1.028	4.283	30	
	40	.2362	.9717	.2431	4.112	1.029	4.232	20	
	50	.2391	.9710	.2462	4.061	1.029	4.182	10	
14	0	.2419	.9703	.2493	4.011	1.031	4.133	0	76
	10	.2447	.9696	.2524	3.961	1.031	4.086	50	
	20	.2476	.9688	.2555	3.913	1.032	4.039	40	
	30	.2504	.9681	.2586	3.866	1.033	3.994	30	
	40	.2532	.9674	.2617	3.820	1.033	3.949	20	
	50	.2560	.9667	.2648	3.776	1.034	3.906	10	
15	0	.2588	.9659	.2679	3.732	1.035	3.864	0	75
	10	.2616	.9651	.2710	3.689	1.036	3.822	50	
	20	.2644	.9644	.2742	3.647	1.036	3.781	40	
	30	.2672	.9636	.2773	3.605	1.037	3.742	30	
	40	.2700	.9628	.2804	3.565	1.038	3.703	20	
	50	.2728	.9620	.2836	3.526	1.039	3.665	10	
16	0	.2756	.9613	.2867	3.487	1.040	3.628	0	74
	10	.2784	.9604	.2899	3.449	1.041	3.591	50	
	20	.2812	.9596	.2931	3.412	1.042	3.556	40	
	30	.2840	.9588	.2962	3.376	1.043	3.521	30	
	40	.2868	.9580	.2994	3.340	1.043	3.487	20	
	50	.2896	.9571	.3025	3.305	1.044	3.453	10	
17	0	.2924	.9563	.3057	3.270	1.046	3.420	0	73
	10	.2951	.9554	.3089	3.237	1.046	3.388	50	
	20	.2979	.9546	.3121	3.204	1.047	3.356	40	
	30	.3007	.9537	.3153	3.171	1.048	3.325	30	
	40	.3035	.9528	.3185	3.139	1.049	3.295	20	
	50	.3062	.9519	.3217	3.108	1.050	3.265	10	
17	68	.3090	.9510	.3249	3.077	1.051	3.236	0	72
		Coſine.	Sines.	Co-Ta.	Tang.	Co-ſec.	Secants		

D.	M.	Sines.	Coſines	Tang.	Co-Ta.	Secants	Co-Se.	M.	D.
18	0	·3090	·9510	·3249	3·077	1·051	3·236	0	72
	10	·3118	·9501	·3281	3·047	1·052	3·207	50	
	20	·3145	·9492	·3314	3·017	1·053	3·179	40	
	30	·3173	·9483	·3346	2·988	1·054	3·151	30	
	40	·3200	·9474	·3378	2·960	1·055	3·124	20	
	50	·3228	·9464	·3410	2 932	1·056	3·098	10	
19	0	·3256	·9455	·3443	2·904	1·058	3·071	0	71
	10	·3283	·9446	·3476	2·877	1·058	3·045	50	
	20	·3310	·9436	·3508	2·850	1·059	3·020	40	
	30	·3338	·9426	·3541	2·824	1·061	2·996	30	
	40	·3365	·9417	·3574	2·798	1·062	2·971	20	
	50	·3393	·9407	·3607	2·772	1 063	2·947	10	
20	0	·3420	·9397	·3640	2·747	1·064	2·924	0	70
	10	·3447	·9387	·3673	2 723	1·065	2·901	50	
	20	·3475	·9377	·3706	2·698	1·066	2·878	40	
	30	·3502	·9367	·3739	2·675	1·069	2·883	30	
	40	·3529	·9356	·3772	2·651	1·069	2·833	20	
	50	·3556	·9346	·3805	2·628	1·070	2·812	10	
21	0	·3584	·9336	·3839	2·605	1·071	2·790	0	69
	10	·3611	·9325	·3872	2 583	1·072	2·769	50	
	20	·3638	·9315	·3905	2·560	1·073	2·749	40	
	30	·3665	·9304	·3939	2·539	1·075	2·728	30	
	40	·3692	·9293	·3973	2·517	1·076	2·708	20	
	50	·3719	·9283	·4006	2·496	1·077	2 689	10	
22	0	·3746	·9272	·4040	2·475	1·078	2·669	0	68
	10	·3773	·9261	·4074	2·454	1·080	2·650	50	
	20	·3800	·9250	·4108	2·434	1·081	2·632	40	
	30	·3827	·9239	·4142	2·414	1·082	2·613	30	
	40	·3854	·9227	·4176	2·394	1·084	2·595	20	
	50	·3880	·9216	·4210	2·375	1·085	2·577	10	
23	0	·3907	·9205	·4245	2·356	1·086	2·559	0	67
	10	·3934	9194	·4279	2·337	1·088	2·542	50	
	20	·3961	·9182	·4313	2·318	1·089	2·525	40	
	30	·3987	·9171	·4348	2·300	1·090	2·508	30	
	40	·4014	·9159	·4383	2·282	1·092	2 491	20	
	50	·4041	·9147	·4417	2 264	1·093	2·475	10	
23	60	·4067	·9135	·4452	2·246	1 095	2·458	0	66
		Coſine.	Sine.	Co-Ta.	Tang.	Co-Se.	Secant.		

D.	M.	Sine.	Coſines	Tang.	Co-Ta.	Secants	Co-Se.	M.	D.
24	0	.4007	.9135	.4452	2.246	1.095	2.458	0	66
	10	.4094	.9123	.4487	2.228	1.096	2.443	50	
	20	.4120	.9112	.4522	2.213	1.097	2.427	40	
	30	.4147	.9100	.4557	2.194	1.099	2.411	30	
	40	.4173	.9087	.4592	2.177	1.100	2.396	20	
	50	.4200	.9075	.4628	2.161	1.102	2.381	10	
25	0	.4226	.9063	.4631	2.144	1.103	2.366	0	65
	10	.4252	.9051	.4698	2.128	1.105	2.351	50	
	20	.4279	.9038	.4734	2.112	1.106	2.337	40	
	30	.4305	.9026	.4770	2.096	1.108	2.323	30	
	40	4331	.9013	.4805	2.081	1.109	2.309	20	
	50	.4357	9001	.4841	2.065	1.111	2.295	10	
26	0	.4384	.8988	.4877	2.050	1.113	2.281	0	64
	10	.4410	.8975	.4913	2.035	1.114	2.268	50	
	20	.4436	.8962	4950	2.020	1.116	2.254	40	
	30	.4462	.8949	.4986	2.006	1.117	2.241	30	
	40	.4488	.8930	.5022	1.991	1.119	2.228	20	
	50	.4514	.8923	.5058	1.977	1.120	2.215	10	
27	0	.4540	.8910	.5095	1.963	1.122	2.203	0	63
	10	.4565	.8897	.5132	1.948	1.124	2.190	50	
	20	.4592	.8883	.5169	1.935	1.126	2.178	40	
	30	.4017	.8870	.5206	1.921	1.127	2.166	30	
	40	.4643	.8856	.5243	1.907	1.129	2.154	20	
	50	.4669	.8843	.5280	1.894	1.131	2.142	10	
28	0	.4695	.8829	.5317	1.881	1.132	2.130	0	62
	10	.4720	.8816	.5354	1.867	1.134	2.118	50	
	20	.4746	.8802	.5392	1.855	1.136	2.107	40	
	30	.4771	.8788	.5429	1.842	1.138	2.096	30	
	40	.4797	.8774	5467	1.829	1.139	2.084	20	
	50	.4823	.8760	.5505	1.816	1.141	2.073	10	
29	0	.4848	.8746	.5543	1.804	1.143	2.063	0	61
	10	.4873	.8732	.5581	1.792	1.145	2.052	50	
	20	.4899	.8718	.5619	1.779	1.147	2.041	40	
	30	4924	.8703	.5658	1.767	1.149	2.030	30	
	40	.4949	.8689	.5696	1.755	1.151	2.020	20	
	50	.4975	.8675	.5735	1.743	1.153	2.010	10	
29	60	.5000	.8660	.5773	1.732	1.155	2.000	0	60
		Coſine.	Sine.	Co-Ta.	Tang.	Co-Se.	Secant.		

D.	M.	Sines.	Coſines	Tang.	Co-Ta.	Secants	Co-Se.	M.	D.
30	0	.5000	.8660	.5773	1.732	1.155	2.000	0	60
	10	.5025	.8645	.5812	1.720	1.156	1.990	50	
	20	.5050	.8631	.5851	1.709	1.158	1.980	40	
	30	.5075	.8616	.5890	1.698	1.160	1.970	30	
	40	.5100	.8601	.5930	1.686	1.162	1.961	20	
	50	.5125	.8586	.5969	1.675	1.165	1.951	10	
31	0	.5150	.8572	.6008	1.664	1.166	1.942	0	59
	10	.5175	.8556	.6048	1 653	1.168	1.932	50	
	20	.5200	.8541	.6088	1.642	1.171	1 923	40	
	30	.5225	.8526	.6128	1.632	1.173	1.914	30	
	40	.5249	.8511	.6168	1.621	1.175	1.905	20	
	50	.5274	.8496	.6208	1.611	1.177	1.896	10	
32	0	.5299	.8480	.6248	1.600	1.179	1.887	0	58
	10	.5324	.8465	.6289	1.590	1.181	1.878	50	
	20	.5348	.8449	.6330	1.580	1.183	1.870	40	
	30	.5373	.8434	.6371	1.570	1.186	1.861	30	
	40	.5397	.8418	.6412	1.559	1.188	1.853	20	
	50	.5422	.8402	.6453	1.550	1.190	1.844	10	
33	0	.5446	.8387	.6494	1.540	1.192	1.836	0	57
	10	.5471	.8371	.6535	1.530	1.195	1.828	50	
	20	.5495	.8355	.6577	1.520	1.197	1.820	40	
	30	.5519	.8339	.6619	1.511	1.199	1.812	30	
	40	.5544	.8323	.6661	1.501	1.201	1.804	20	
	50	.5568	.8306	6703	1.492	1.203	1.796	10	
34	0	.5592	.8290	.6745	1.482	1.206	1.788	0	56
	10	.5616	.8274	6787	1.473	1.208	1.780	50	
	20	.5640	.8258	.6830	1.464	1.211	1.773	40	
	30	.5664	.8241	.6873	1.455	1.213	1.765	30	
	40	.5688	.8225	.6916	1.446	1.216	1.758	20	
	50	.5712	.8208	.6958	1.437	1.218	1.751	10	
35	0	.5736	.8191	.7002	1.428	1.220	1.743	0	55
	10	.5759	.8175	.7045	1.419	1.223	1.736	50	
	20	.5783	.8158	.7089	1.411	1.226	1.729	40	
	30	.5807	.8141	.7133	1.402	1.228	1.722	30	
	40	.5831	.8124	.7177	1.393	1.231	1.715	20	
	50	.5854	.8107	.7221	1.385	1.233	1.708	10	
35	60	.5878	.8090	.7265	1.376	1.236	1.701	0	54
		Coſine.	Sine.	Co Ta.	Tang.	Co-Se.	Secant.		

D.	M.	Sines.	Cofines	Tang.	Co-Ta.	Secants	Co-fec.	M.	D.
36	0	.5878	.8090	.7265	1.376	1.236	1.701	0	54
	10	.5901	.8073	.7310	1.368	1.239	1.694	50	
	20	.5925	.8056	.7355	1.359	1.241	1.688	40	
	30	.5948	.8038	.7400	1.351	1.244	1.681	30	
	40	.5971	.8021	.7445	1.343	1.247	1.674	20	
	50	.5995	.8004	.7490	1.335	1.249	1.668	10	
37	0	.6018	.7986	.7535	1.327	1.252	1.662	0	53
	10	.6041	.7969	.7581	1.319	1.255	1.655	50	
	20	.6064	.7951	.7627	1.311	1.258	1.649	40	
	30	.6088	.7933	.7673	1.303	1.260	1 643	30	
	40	.6111	.7916	.7719	1.295	1.263	1.636	20	
	50	.6134	.7898	.7766	1.287	1.266	1.630	10	
38	0	.6157	.7880	.7813	1.280	1.269	1.624	0	52
	10	.6179	.7862	.7860	1.272	1.272	1.618	50	
	20	.6202	.7844	.7907	1.265	1.275	1.612	40	
	30	.6225	.7826	.7954	1.257	1.278	1.606	30	
	40	.6248	.7808	.8002	1.249	1.281	1.600	20	
	50	.6270	.7790	.8050	1.242	1.284	1.595	10	
39	0	.6293	.7771	.8098	1.235	1.287	1 589	0	51
	10	.6316	.7753	.8146	1.227	1.290	1.583	50	
	20	.6338	.7735	.8195	1.220	1.293	1.578	40	
	30	.6361	.7716	.8243	1.213	1.296	1.572	30	
	40	.6383	.7698	.8292	1.206	1.299	1.567	20	
	50	.6405	.7679	.8341	1.199	1.302	1 561	10	
40	0	.6428	.7660	.8391	1.192	1.305	1.556	0	50
	10	.6450	.7642	.8441	1.185	1.309	1.550	50	
	20	.6472	.7623	.8491	1.178	1.312	1 545	40	
	30	.6494	.7604	.8541	1.171	1.315	1.540	30	
	40	.6516	.7585	.8591	1.164	1.318	1.534	20	
	50	.6539	.7566	.8642	1.157	1.322	1.529	10	
41	0	.6560	.7547	.8693	1.150	1.325	1.524	0	49
	10	.6582	.7528	.8744	1.143	1.328	1.519	50	
	20	.6604	.7509	.8795	1.137	1.332	1.514	40	
	30	.6626	.7489	.8847	1.130	1.335	1.509	30	
	40	.6648	.7470	.8899	1.124	1.339	1.504	20	
	50	.6670	.6670	.8951	1.117	1.342	1.499	10	
41	60	.6691	.7431	.9004	1.111	1.346	1.494	0	48
		Cofine.	Sine.	Co-Ta.	Tang.	Co-fec.	Secant.		

D.	M.	Sines.	Coſines	Tang.	Co-Ta.	Secants	Co-Se.	M.	D.
42	0	.6691	.7431	.9004	1.111	1.346	1.494	0	48
	10	.6713	.7412	.9057	1.104	1.349	1.490	50	
	20	.6734	.7392	.9020	1.098	1.353	1.485	40	
	30	.6756	.7373	.9163	1.091	1.356	1.480	30	
	40	.6773	.7353	.9217	1.085	1.360	1.475	20	
	50	.6799	.7333	.9271	1.079	1.364	1.471	10	
43	0	.6820	.7313	.9325	1.072	1.367	1.466	0	47
	10	.6841	.7294	.9380	1.066	1.371	1.462	50	
	20	.6862	.7273	.9434	1.060	1.375	1.457	40	
	30	.6883	.7254	.9490	1.054	1.379	1.453	30	
	40	.6905	.7234	.9545	1.048	1.382	1.448	20	
	50	.6926	.7213	.9601	1.041	1.386	1.444	10	
44	0	.6946	.7193	.9657	1.035	1.390	1.439	0	46
	10	.6967	.7173	.9713	1.029	1.394	1.435	50	
	20	.6988	.7153	.9770	1.023	1.398	1.431	40	
	30	.7009	.7132	.9827	1.018	1.402	1.427	30	
	40	.7030	.7112	.9884	1.012	1.406	1.422	20	
	50	.7050	.7092	.9942	1.006	1.410	1.418	10	
44	60	.7071	.7071	1.000	1.000	1.414	1.414	0	45
		Coſine.	Sine.	Co-Ta.	Tang.	Co-Se.	Secant.		

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In the *Eſſay* on *Geometry.*

P. 6. l. 7. *b.* for *and* r. *a.* P. 23. l. 11. in the Margin, add *Fig.* 8 and 9. P. 28. l. 2. *b.* for $x = rr$. r. x, rr. P. 29. l. 1. *b.* for IK r. IH; and l. 4. *b.* for IK r. IH; ſame Page, in the Note, for $-am+m^2$ r. $-am-m^2$. P. 30. l. 9. for HD r. HDB. P. 33. l. 9. for DEF r. DEH. P. 34. l. 2. in the Margin, for 58 r. 92. P. 35. l. 11. for BF r. BK. P. 37. l. 11. for AB r. AD. P. 42. l. 4. *b.* to *ſaid* $\angle DBA$ make a Reference, 147. P. 51. the * in the Note refers to Art. 111. P. 52. l. 16. for DC r. DE. P. 53. l. 17. for BE r. BC. P. 55. l. 13. for CB r. CF. P. 56. l. 7. for D*f* r. DF. P. 58. l. 1. *b.* for BC r. BE; and l. 9. *b.* for BD r. BE. P. 63. l. 11. *b.* for $BC \times DE$ r. $BC \times DC$. P. 80. l. 5. for FG r. FH. P. 81. l. 3. for *other Leg* r. *other Ruler.* P. 83. l. 2. for $\triangle EBG$ r. $\angle EBG$; *ibid.* l. 18. for $\angle EBG$ r. $\angle D$; and *ibid.* l. 10. *b.* for HFK r. HKF. P. 86. l. 8. for *it* r. BF. P. 90. l. 16. for $\angle KBA = \angle DEF$ r. $\angle BKA = \angle DEG$. P. 91. l. 9. *b.* for $\perp$ *to it the Diameter* AC r. $\perp$ *to it the Diameter* BD. P. 93. l. 15. for GBA r. GBE. P. 111. l. 3. for $DE^2=225$ r. $DC^2=225$; and for $DC^2=325$ r. $DF^2=325$; and for 28)125 r. 28)225. P. 131. dele *alſo for* 7632407 r. 7629.4317.

N. B. b ſignifies to count from the Bottom of the Page.

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The ſeveral Parts are to be bound in the following Order.

1. The general Title and Preface.
2. The Eſſay on Logarithms.
3. ———————— Geometry.
4. ———————— Trigonometry.

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The ſix Plates of Trigonometry may be placed at the End.

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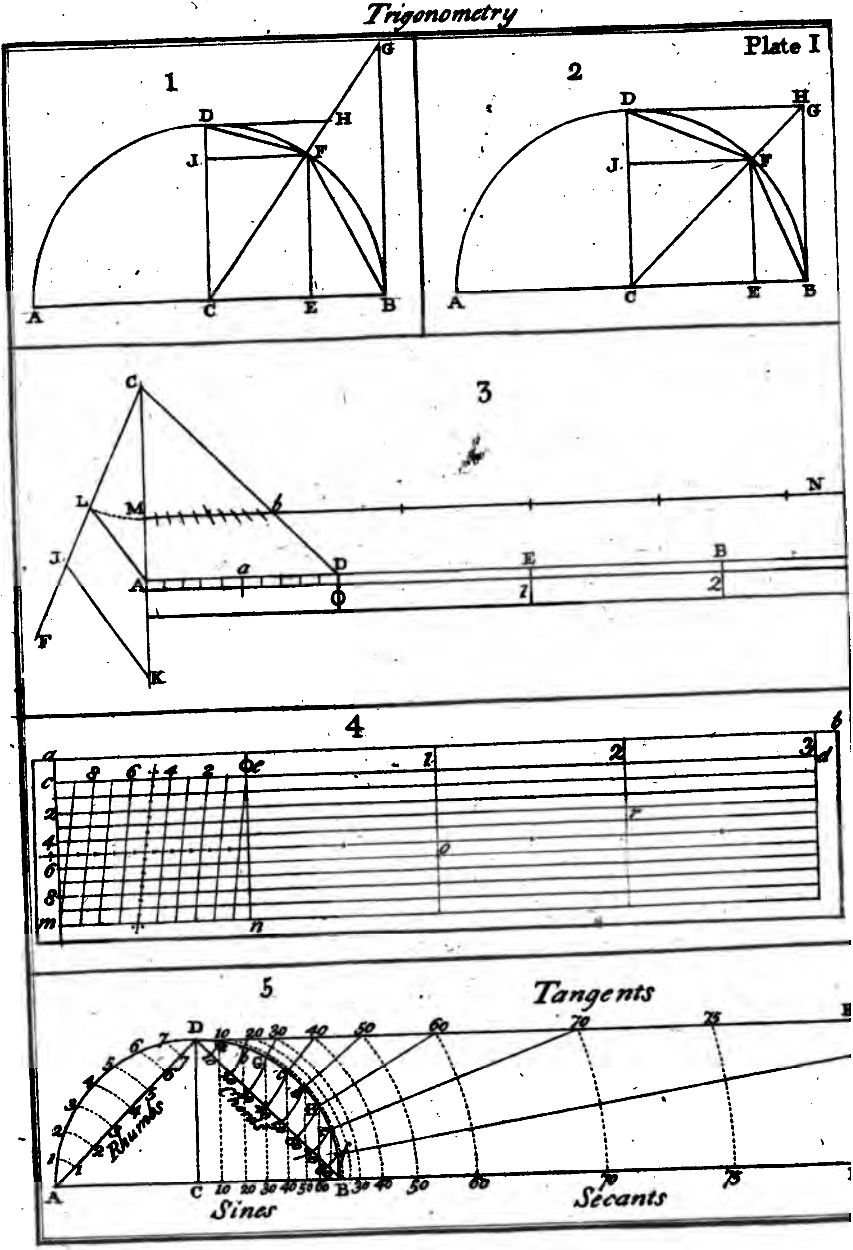

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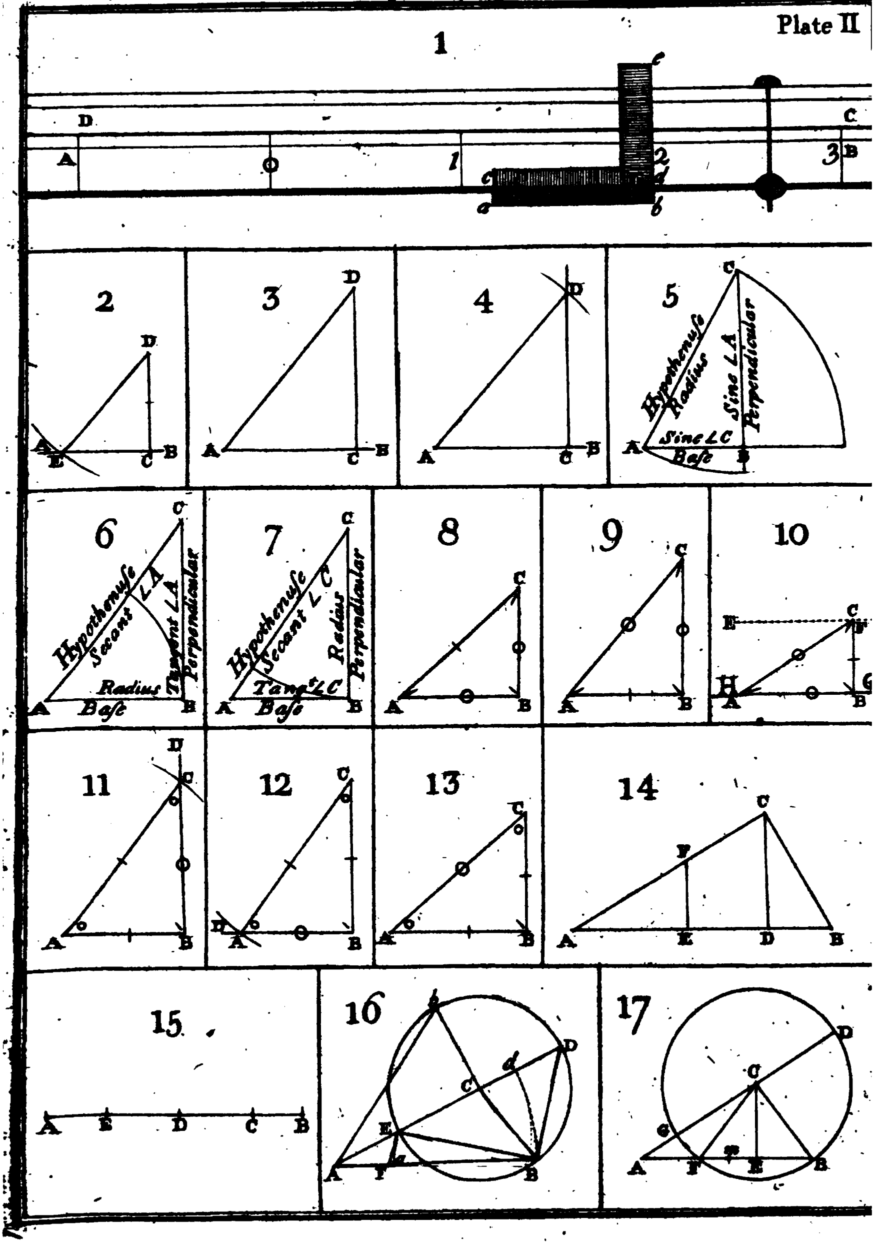
Trigonometry

Plate II



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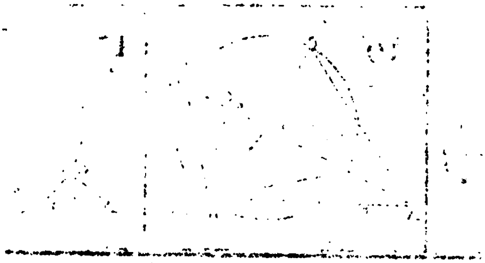

Trigonometry

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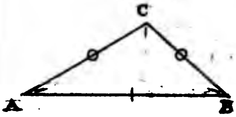

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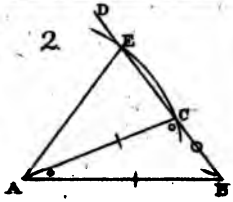

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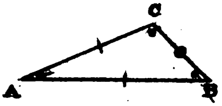

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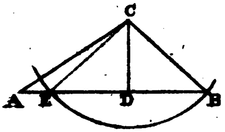

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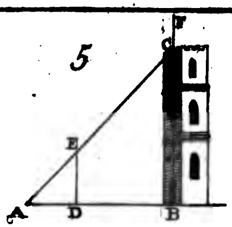

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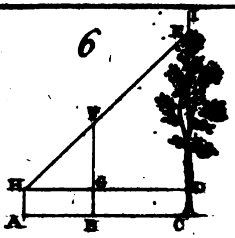

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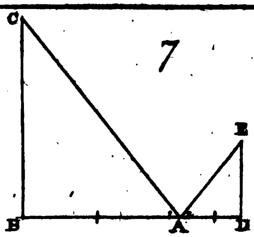

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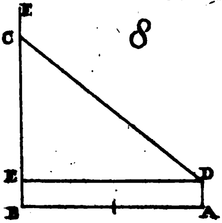

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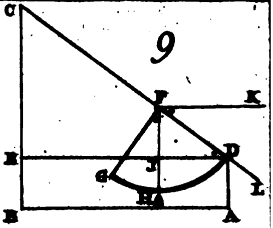

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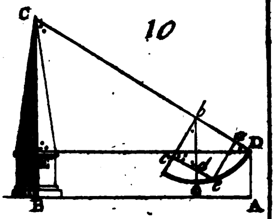

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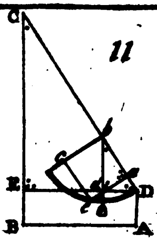

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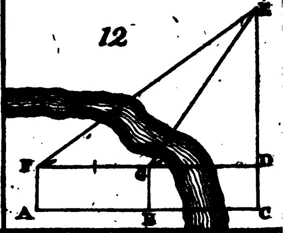

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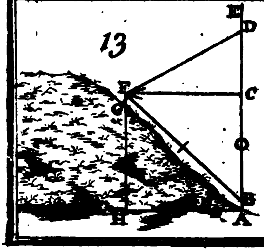

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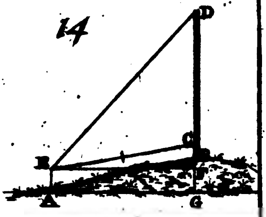

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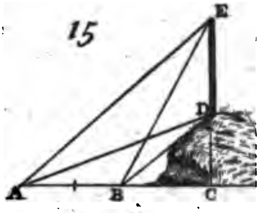

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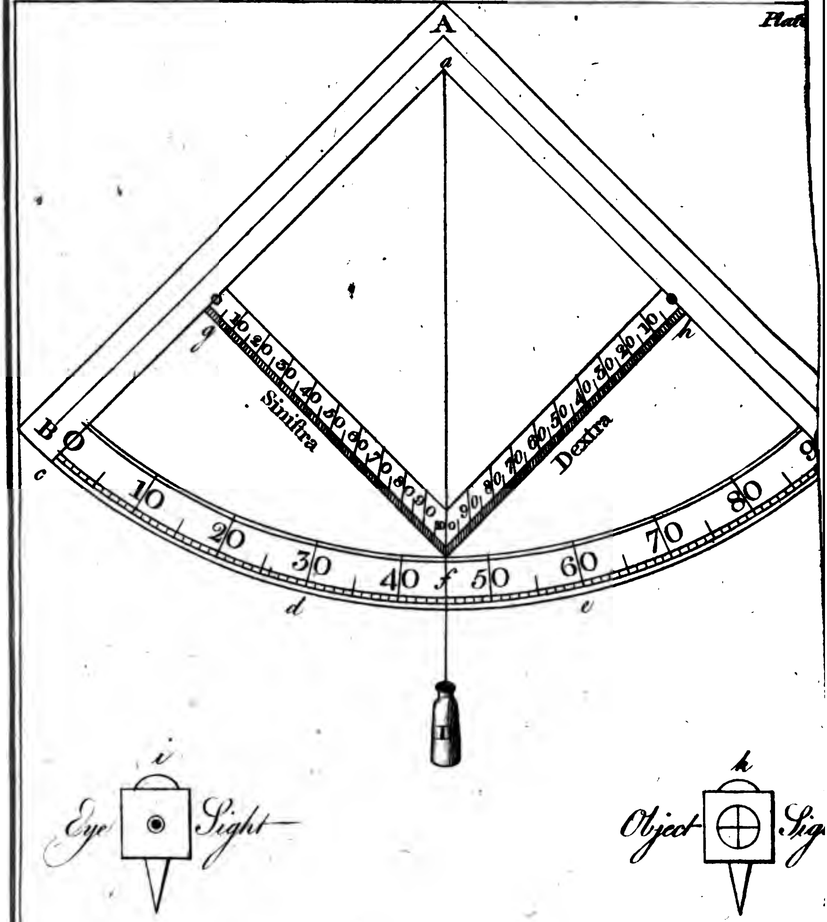
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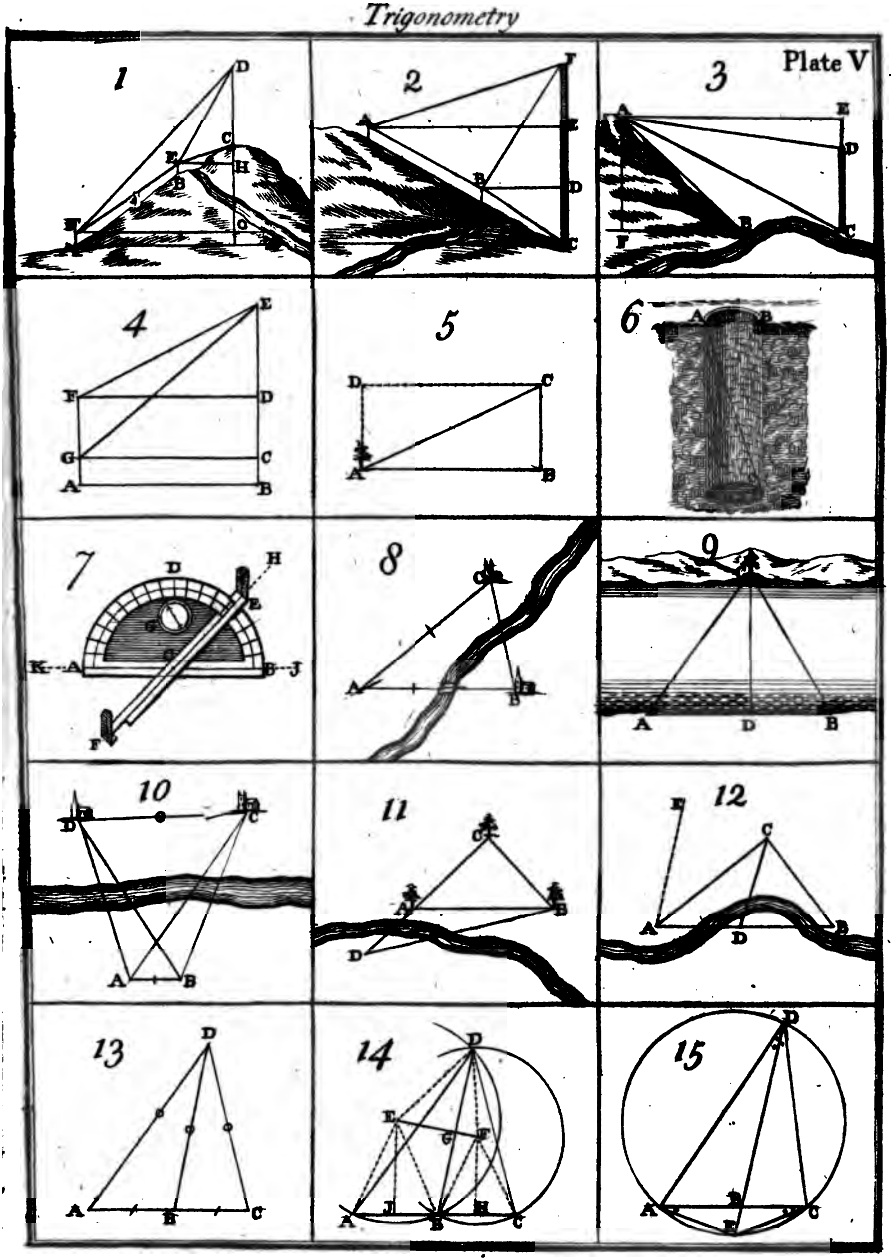
Trigonometry

Plate V



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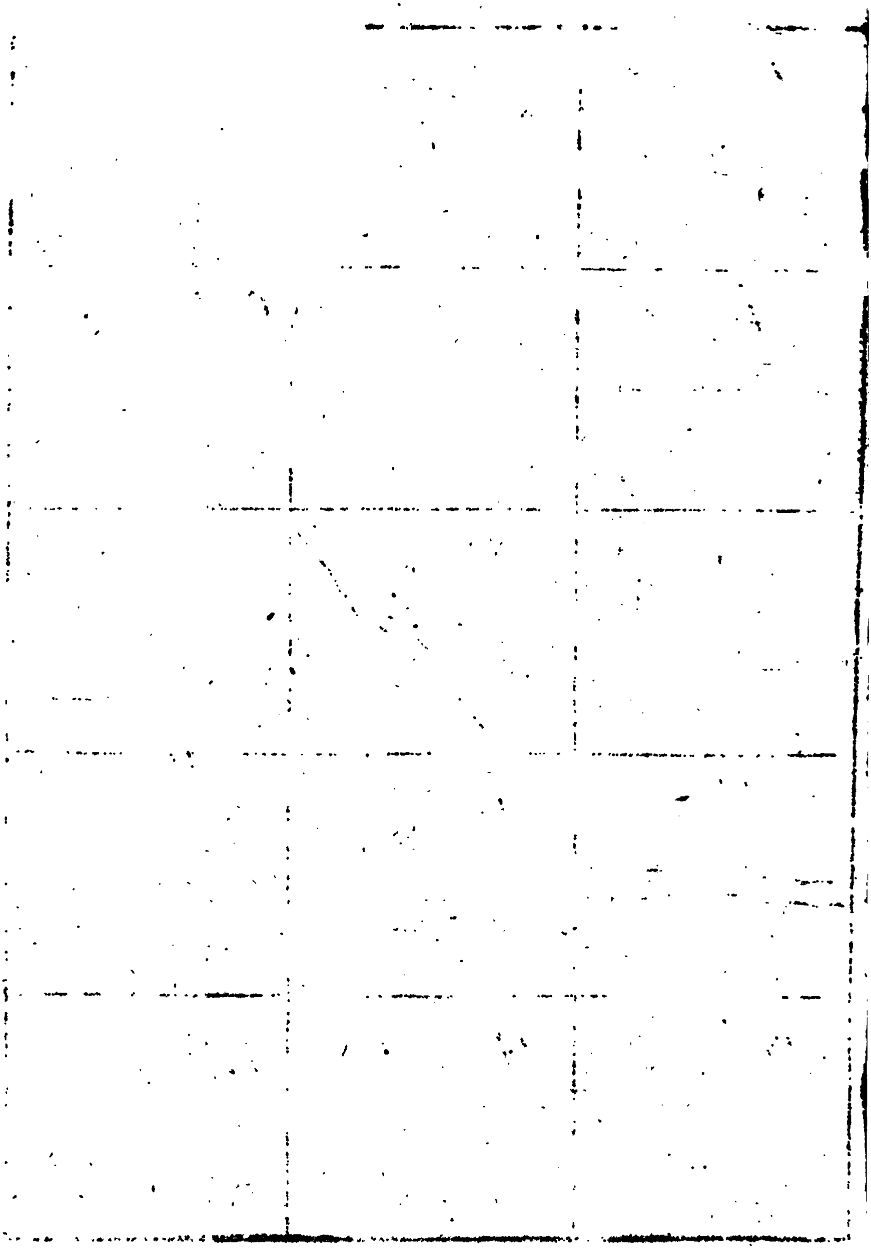
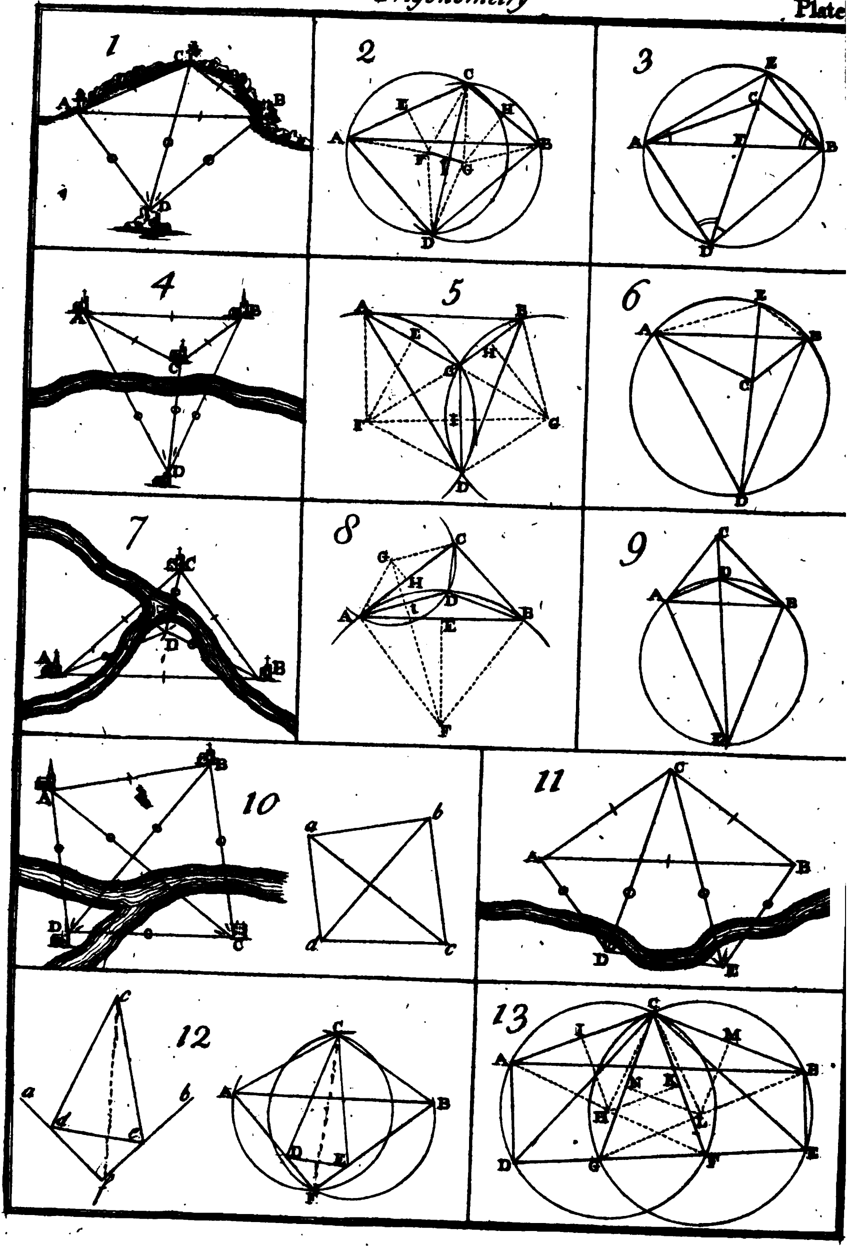


Trigonometry **Plate**



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